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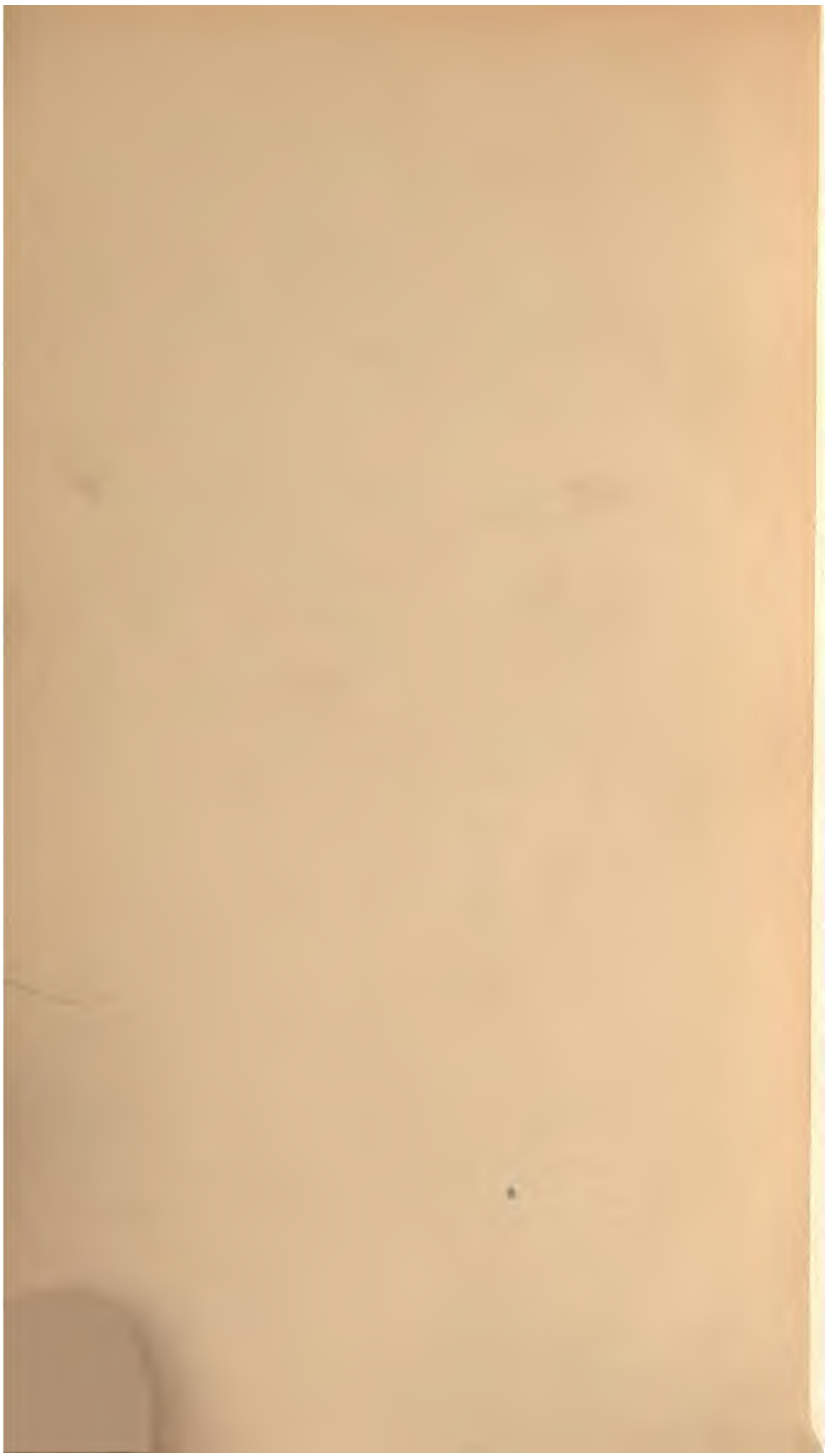

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and

$$\int_0^1 (x^p + x^{-p}) \log(1-x) \frac{dx}{x} = \frac{\pi}{p} \cot p\pi - \frac{1}{p^2} \ldots\ldots(2),$$

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5624. (Prof. Sylvester, F.R.S.)—If X_j is to be a rational homogeneous function of the j^{th} order of i independent quantities $a, b, c, \ldots k, l$, and is to satisfy the partial differential equation

$$(A\delta_a + B\delta_b + \ldots + L\delta_l)\, X_j = 0,$$

where A, B, ... L are homogeneous linear functions of $a, b, \ldots l$, not absolutely independent, but connected by a single homogeneous linear equation with one another; prove that the number of arbitrary parameters in X_j will in general be the number of solutions in positive integers (zero counted in as positive) of the simultaneous equations

$$u_0 + u_1 + u_2 + \ldots + u_{i-1} = j\tfrac{1}{2}v_0 + u_1 + 2u_2 + \ldots (i-1)\, u_{i-1} = (i-1)\tfrac{1}{2}jv_0 + v_1 = 1.$$

Or, again, suppose $[(a-b)\,\delta_c + (b-c)\,\delta_a + (c-a)\,\delta_b]\, X_2 = 0$, the complete value of X_2, viz., $k(a^2+b^2+c^2) + k'(ab+ac+bc)$, contains 2 parameters k, k', which will be found to agree with the rule; and so, if in the above we write X_{2i} or X_{2i+1} in place of X_2, it will be found (still in accordance with the rule) that there will be $i+1$ arbitrary parameters in each of these forms. 55

5708. (Prof. Seitz, M.A.)—Within a given circle a line is drawn at random in position and length; show that (1) the chance that the line intersects a given diameter is π^{-1}; and (2) the average length of the line is $\frac{3}{16}\pi r$. 107

5791. (W. H. H. Hudson, M.A.)—Let AB, BC, CD be equal arcs of a circle, TA, TD be the tangents at A, D; AB, DC meet in Y; XB, XC, the tangents at B, C, meet AD, produced in EF; show that, when B, C, D move up to A, the ultimate ratio of the triangles TAD, XEF, YAD, are as 27 : 25 : 18. 30

5889. (J. J. Walker, M.A.) — Show that $l\alpha + m\beta + n\gamma$ will be a side of the cone having origin as vertex, and the circle through the terms of α, β, γ as base, if

$$l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha) + n^{-1}T^2(\alpha-\beta) = 0;$$

and hence deduce immediately the existence of the second system of cirular sections parallel to the plane through the terms of α^{-1}, β^{-1}, γ^{-1}. 61

5901. (Prof. Sylvester, F.R.S.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu - \mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu' - \mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it..... 34

5926. (Prof. Sylvester, F.R.S.) — By the potential work (in respect to an arbitrary centre) of a body describing any orbit, plane or twisted, understand what the work would be were it moving in a circle about such centre under the influence of the

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forces actually soliciting it; and by the residual work understand the excess of the actual stored-up work over such potential work. Also in general let an acceleration in any quantity (q) signify its second time-derivative $\left(\frac{d^2q}{dt^2}\right)$. Prove that the residual work of a particle of unit mass at each moment of time, in respect to any centre, is equal to the acceleration in the square of the radius vector drawn to such centre..... 65

5947. (Prof. Seitz, M.A.)—Two points are taken at random within a circle, one on each side of a chord which divides the circle into two segments whose areas are u_1, u_2; show that the chance that the chord drawn through the points is less than a chord which cuts an area u from the circle, is

$$\frac{u^2}{u_1 u_2}, \text{ when } u < u_1 < u_2, \text{ and } \frac{2u-u_1}{u_2}, \text{ when } u > u_1. \text{ .....} \quad 59$$

5962. (Prof. Cochez.) — Trouver la courbe dont le rapport du rayon de courbure à la normale est constant et égal à m. 72

5968. (The late Prof. Benjamin Peirce, F.R.S.) — If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction..... 111

5973 & 6002. (J. L. Mackenzie, B.A.)—Solve the functional equation $\phi(ax+\beta) = \gamma\phi(x) + u$, where α, β, γ are constants and u is a known function of x; and, as particular cases, solve the equations

$$\phi(a-x)\phi(x) + b[\phi(a-x) + \phi(x)] = c^2, \ \phi(mx) = \phi(x) + ax\log x. \quad 28$$

5976. (R. Tucker, M.A.) — P_1, P_2, P_3; (Q_1, Q'_1), (Q_2, Q'_2), (Q_3, Q'_3) are the respective denominators and numerators of the ultimate (proper fraction) convergents, which are obtained from making the roots of the equation $x^3 + qx + r = 0$ the quotients of a series of continued fractions; shew that

$$\frac{Q_1Q'_1}{P_1} + \frac{Q_2Q'_2}{P_2} + \frac{Q_3Q'_3}{P_3} = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3}. \text{ .....} \quad 69$$

5983. (Prof. Sylvester, F.R S.)—The i^{th} involute to a circle being defined as a cyclode of the i^{th} order, and a symmetrical cyclode as one which can be divided into two precisely equal and similar parts; prove that there are as many distinct species of symmetrical cyclodes of the order $2n$ possessing the property that the length of the arc of any one of them is connected by an algebraical equation with the length of the corresponding arc of its pedal, as there are ways of breaking up the number n into parts all unequal to one another..... 79

6000. (W. S. B. Woolhouse, F.R.A.S.)—The separate probabilities of three partially dependent events are given as well as the probabilities of the joint occurrence of every two events and also the joint occurrence of all three. In a proposed case, however, it is known that two of the events have failed, but it is unknown which; determine the probability of the failure also of the remaining event..... 26

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6005. (The Editor.)—If three lengths are taken at random, and the limits be the same (and any magnitude) for all three, show that the respective probabilities of being able to form with them (1) a triangle of any kind, (2) an obtuse-angled triangle, are

$$p_1 = \tfrac{1}{2},\ p_2 = \tfrac{1}{4}\pi \ldots\ldots 22$$

6008. (Prof. Sylvester, F.R.S.)—Prove that the equation

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = 0$$

is insoluble by positive or negative integer values of x, y, z..... 99

6019 & 6088. (T. R. Terry, M.A.)—If $a < 1$ and m and n are zero or any positive integers, and

$$f(n,m) = \frac{(2n)!}{2^{2n}n!\,n!}\left\{1 - \frac{n-m}{n+1}na^2 + \frac{(n-m)(n-m+1)}{n+1\,.\,n+2}\cdot\frac{m(m-1)}{1\,.\,2}a^4 - \ldots\right\},$$

prove that $4a^2f(n,m) = -f(n-1,m+1) + 2(1+a^2)f(n-1,m)$

$$-(1-a^2)^2 f(n-1,m-1) \ldots\ldots (1),$$

$$\frac{df(n,m)}{da} = 4a\frac{(n-m)(n-m+1)}{2n+1}f(n+1,m-1) - 2a(n-m)f(n,m-1) \ldots (2),$$

$$f(n,m+1) = (1+a^2)f(n,m) - \frac{4a(n-m)}{2n+1}f(n+1,m) \ldots\ldots (3).\quad 27$$

6034. (Prof. Sylvester, F.R.S.)—

$$\left.\begin{array}{llll} U_1, & U_2, & \ldots & U_i \\ V_1, & V_2, & \ldots & V_j \\ W_1, & W_2, & \ldots & W_k \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right\}$$

are homogeneous numerical linear functions of n variables; and $i+j+k+\ldots = n$. Any i homogeneous linear functions, in which the coefficients are all integer, are taken of the U's, j of the V's, k of the W's, and so on. Let R be the resultant of these n new linear functions of the n variables. Investigate a rule for determining the greatest common measure of the infinite number of values of R that can be thus obtained 71

6037. (Prof. Crofton, F.R.S.)—Prove that—

$e^{x+D}\,F(x) = e^1 e^x\,F(x+1)$, and $e^{a(x+D)}\,Fx = e^{1a^2}e^{ax}\,F(x+a)$.. 41

6066. (Prof. Wolstenholme, M.A.) — A heavy particle moves on the interior of a smooth paraboloid whose axis is vertical, and z_1, z_2 are its least and greatest vertical heights above the vertex, ρ the radius of absolute curvature of its path when at a vertical height z: prove that

$$\frac{4a(a+z)^3}{z}\cdot\frac{(z_1+z_2-z)^2}{\rho^2} = \frac{\{(z_1+z_2-z)[a(z_1+z_2-z)+z_1z_2]^2 + az_1z_2(a+z)^2\}^2}{z(z_1+z_2-z)^2[a(z_1+z_2-z)+z_1z_2]^2 + az_1^2z_2^2(a+z)^2}.$$

..... 88

6072. (Prof. Cochez.)—Sur les côtés d'un triangle et dans le même sens on prend à partir des sommets de longueurs AD, BE, CF égales respectivement à $\frac{m\text{AB}}{n}$, $\frac{m\text{BC}}{n}$, $\frac{m\text{AC}}{n}$, et on joint les points D, E, F. Déterminer (1) le rapport $\frac{\text{DEF}}{\text{ABC}}$; (2) le rapport $\frac{a^2+b^2+c^2}{a'^2+b'^2+c'^2}$, a, b, c étant les côtés de ABC, et a', b', c' les côtés de DEF; (3) comment faut-il déterminer le rapport $\frac{m}{n}$ pour que le triangle DEF ait une surface minima ? 93

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6075. (Donald McAlister, D.Sc.)—A sphere full of water, is hung up by a string. Suddenly a slender uniform crack extends from one end of the vertical diameter to the other. Prove that the time of emptying (on the hypothesis of "parallel sections") is to the time of free fall from rest down the diameter, as the area of the sphere is to that of the lune. 36

6089. (D. Edwardes.)—If A, B, C are the angles of a triangle, prove that

$$2\{\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A)+\sin(A-B)\sin(C-A)\cos A$$
$$+\sin(B-C)\sin(A-B)\cos B+\sin(C-A)\sin(B-C)\cos C\}$$
$$=(\sin^2 A+\sin^2 B+\sin^2 C)(1-8\cos A\cos B\cos C)$$ 93

6091. (R. Tucker, M.A.) — If a point be taken on a certain line and normals be drawn from it to an ellipse, a circle can be drawn through the intersections of the normals with the ellipse. Find the envelop of this circle..... 85

6094. (Prof. Sylvester, F.R.S.) — 1. If the process of residuation, described in Question 2391, be applied to a quartic instead of to a cubic curve, prove that the minimum number of points in any *residuum* is 3, that being the number applicable to the case of an odd number of initial points. If the initial number is of the form $4i$, show that the minimum number in a residuum is 4, and when the initial number is of the form $4i+2$, 6, except for the case when $i=0$, when it is obviously only 2.

2. Prove that, if the number of initial points is odd, the 3 *residuals* are functions only of those points.

3. In general, if p be prime to μ, and p initial points be taken in a curve of the μ^{th} order, prove that the residuum may be reduced to consist of $\frac{1}{2}(\mu-1)(\mu-2)$ residual points, which residuals are all functions of the p given points only, being independent of the particular process of residuation applied to arrive at them..... 34

6101. (J. J. Walker, M.A.)—If one side of a thin bi-convex lens is turned towards the sun, a well-defined image may be formed on a small screen facing that side, say at a distance of α^{-1}. The same being done with the other side, and the distance being now α'^{-1}, show that, if μ is the refractive index, the power of the lens is given by

$$2(2\mu-1)\phi=-(\mu-1)(\alpha+\alpha')$$ 88

6106. (A. Martin, M.A.)—Show that the mean value of all the radius-vectors that can be drawn from one end of the major axis of a given ellipse to its circumference is $\frac{4b}{\pi e}\sin^{-1}e$..... 70

6128. (Prof. Tanner, M.A.)—The continued fraction

$$\frac{a}{b-}\,\frac{a}{b-}\cdots\frac{a}{b-}\,\frac{a}{b+x}$$ (r terms) being represented by u_r, find a, b so that, for all values of r, $u_r=u_{r+n}$, and verify the result by showing the form of u_r..... 65

6145. (C. Leudesdorf, M.A.)—Find three square numbers such that, if each be subtracted from the sum of the other two, and the remainders doubled, the resulting numbers shall be perfect squares..... 95

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6151. (D. Edwards.)—A ring of weight W is just capable of sliding upon a rough semi-elliptic wire, and has attached to it two strings which, passing through smooth small rings on the fixed support, sustain weights P. If 2θ be the angle between the strings in a symmetrical position of equilibrium, prove that $W^2 \sin^2\theta - \mu^2 W \sin\theta - 2P^2 \cos\theta^2 = \mu^2 W^2$, where μ is the coefficient of friction, and e eccentricity of ellipse. 64

6152. (G. Tarrif, M.A.)—If P, Q be two points on an ellipse such that the difference of their eccentric angles is constant, and if the ordinates at P and Q be produced to meet the auxiliary circle in P' and Q': show (1) that area of $\Delta P'CQ'$: area of $\Delta PCQ = a : b$, C being the centre; (2) that PQ touches at its middle point the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\frac{1}{2}\alpha$. 58

6154. (Prof. Sylvester, F.R.S.)—If the roots of a Quantic are all real, prove that the roots of its Hessian are all imaginary 108

6158. (Prof. Crofton, F.R.S.)—Find the mean distance of two points in a rectangle 101

6169. (Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre: find the average volume of the tetrahedron OPQR. 33

6173. (Hugh McColl, B.A.)—If the statements a, b, f are true, then the statements c and x are true, or else the statements d, e, y are true. If the statement a is true, f false, and y true, then c is true and x false, or else d is false and e true. What inference may be drawn from these premises with respect to the truth or falsehood of the statements a, b, c, d, e, f, eliminating the statements x and y? 40

6186. (T. R. Terry, M.A.)—If $\alpha < 1$, and m and n are any positive integers, prove that

$$1+\frac{(m-1)\,m}{1\,.\,n+1}\frac{\alpha^2}{1-\alpha^2}+\frac{(m-2)(m-1)\,m\,(m+1)}{1\,.\,2\,.\,(n+1)\,(n+2)}\frac{\alpha^4}{(1-\alpha^2)^2}+\ldots=\frac{1}{(1-\alpha^2)^{m-1}}$$
$$\times\left\{1-\frac{n-m+1}{n+1}(m-1)\,\alpha^2+\frac{(n-m+1)\,(n-m+2)}{(n+1)\,(n+2)}\cdot\frac{(m-1)\,(m-2)}{1\,.\,2}\,\alpha^4-\ldots\right\}.$$
..... 104

6188. (Prof. Sylvester, F.R.S.)—If in any persymmetrical matrix of the n^{th} order, each column in succession calling its elements $c_1, c_2 \ldots c_n$, be replaced by $0, c_1, 2c_2 \ldots (n-1)\,c_n$, show that the sum of the n determinants thus formed is zero. For example, show that

$$\begin{vmatrix} 0 & b & c & d \\ a & c & d & e \\ 2b & d & e & f \\ 3c & e & f & g \end{vmatrix} + \begin{vmatrix} a & 0 & c & d \\ b & b & d & e \\ c & 2c & e & f \\ d & 3d & f & g \end{vmatrix} + \begin{vmatrix} a & b & 0 & d \\ b & c & c & e \\ c & d & 2d & f \\ d & e & 3e & g \end{vmatrix} + \begin{vmatrix} a & b & c & 0 \\ b & c & d & d \\ c & d & e & 2e \\ d & e & f & 3f \end{vmatrix} = 0.$$
..... 55

6196. (The Editor.)—If ABC, A'B'C' be two triangles with their corresponding sides equal and parallel; prove that (1) of the parallelograms formed by joining corresponding vertices one is equal to the other two; and hence (2) prove Euc. I. 47..... 37

6203. (W. H. H. Hudson, M.A.)—If P be the population and W the wealth of an industrial community at a time t, (1) interpret $\frac{d}{dt}\left(\frac{W}{P}\right)$; and (2) show that, in such a community, if each

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person saves one-nth of his income, the average value of a person's work per unit of time is

$$n\frac{d}{dt}\left(\frac{W}{P}\right)+\frac{W}{P^2}\frac{dP}{dt} \quad \ldots\ldots 92$$

6204. (J. J. Walker, M.A.)—Referring to Quest. 5680, show that the conditions for the double contact of two conics — not expressible in terms of the invariants of the system—may be obtained in terms of coefficients of the three contravariant conics of the system. 74

6205. (W. J. C. Sharp, M.A.)—Prove that, according as the polar conic of a point on a cubic is an hyperbola, a parabola, or an ellipse, two lines, one line, or no line, can be drawn through the point, so as to be terminated by the curve and bisected at the point. 29

6207. (E. Anthony, M.A.)—In a plane triangle ABC, if $\frac{1}{4}\Sigma\sin^2 A \equiv \sigma$, and $\sigma-\sin^2 A = \sigma_1$, &c., prove that

$$\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2 = \sin^2 A \sin^2 B \sin^2 C. \quad \ldots\ldots 56$$

6211. (R. E. Riley, B.A.)—Two points are taken on the axis of a parabola at equal distances on opposite sides of the focus, and from these points perpendiculars are drawn to a tangent. If SY be the perpendicular from the focus, show that the difference of the other two perpendiculars varies inversely as SY. 60

6212. (D. Edwardes.)—If ρ, R be the radii of absolute and spherical curvature at an ordinary point on a curve in space, and if ds, ds' be the elements of arcs at corresponding points on the locus of the centres of absolute and spherical curvature respectively, prove that $ds : ds' = d\rho : dR$. 62

6213. (A. Martin, M.A.) — A cylinder, of radius r, *rolls* down the surface of another one, radius R, resting on a horizontal plane, the surface of both cylinders and plane being rough enough to secure perfect rolling. Determine the circumstances of the motion, the point of separation, and the path of the axis of the upper cylinder. 44

6214. (G. Heppel, M.A.) — If the sides of a plane triangle be circular arcs of equal radius, find a simple necessary relation between the sides and angles of the triangles. 58

6216. (R. Tucker, M.A.) — O is the point of projection of two equi-horizontal-range particles, and (OP, OP'), (OQ, OQ') the normal chords and diameters of curvature at O respectively; prove that, if the initial velocity be the same for both, PQ' is parallel to QP'. 56

6218. (Prof. Sylvester, F.R.S.)—If

$$E = a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e\ldots, \quad F = a\delta_c+3b\delta_d+6c\delta_e+10d\delta_f\ldots,$$

$$G = a\delta_d+4b\delta_e+10c\delta_f+20d\delta_g\ldots, \quad H = a\delta_e+5b\delta_f+15c\delta_g+35e\delta_h\ldots,$$

express $(E)^n$ as an algebraical function of E, F, G, H, &c. 99

6219. (Prof. Townsend, F.R.S.)—A thin uniform spherical shell being supposed to hold in free equilibrium a material particle, situated at its centre of form, by the attraction of its mass for a single power, direct or inverse, of the distance; determine the groups of powers for which the equilibrium of the particle is respectively stable and unstable. 114

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6226. (Prof. Matz, M.A.)—A post is a feet from the north wall, and b feet from the west wall of a room whose sides meet each other at right angles; find the sides of the largest rectangular table that can be passed so as to touch at the same time the walls and the post, the surface of the table to remain horizontal all the time..... 39

6232. (J. J. Walker, M.A.)—A system of forces being reduced to two, one of which passes through a fixed point and makes a given angle with the resultant of translation; show that the other has for its envelop an ellipse in the plane through the fixed point perpendicular to the axis of principal moment for that point; and that the ellipse becomes a circle when the fixed point is Poinsot's centre. 25

6235. (H. McColl, B.A.) — When, from the three implications, (1) $a'b + ab' : dx$, (2) $ax + by : c$, (3) $cd : y$, may we conclude that either x or y is true, but not both? 69

6238. (R. Knowles B.A., L.C.P.)—A paraboloid, of parameter m, when prevented from sliding down a plane whose inclination is α, just stands on its base without falling over; prove that the length of its axis is $9m \cot^3 \alpha$. 63

6241. (E. W. Symons, B.A.)—The roots of the equation

$$x^3 + \lambda x = (x^2 + 1)(2\lambda + \mu)^{\frac{1}{2}}$$

are all real and positive; prove that neither λ nor μ can be less than 9. 24

6243. (Prof. Sylvester, F.R.S.)—If $2f^2 + g^2$ is a prime and f an odd number, prove that $fy^2 + 2gxy - 2fx^2 = \pm 1$ is soluble in integers. 21

6245. (Prof. Cochez.)—Démontrer que l'enveloppe des axes des coniques tangentes à deux droites en deux points donnés est une parabole..... 49

6246. (Prof. Townsend, F.R.S.)—In the rotation, round its axis of figure, of a liquid spheroid of uniform density, held in permanent form by its own attraction for the ordinary law of the inverse square of the distance; show, from the equation of permanence, that

$$\frac{\omega^2}{4\pi\rho} = \frac{2}{3 \times 5}\lambda^2 - \frac{4}{5 \times 7}\lambda^4 + \frac{6}{7 \times 9}\lambda^6 - \frac{8}{9 \times 11}\lambda^8 + \&c.,$$

the several quantities involved having their usual significations. 30

6248. (The Editor.)—If two points P, Q be taken at random on the area of a vertical circle; show that the probability that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{6}$ 111

6249. (Prof. Matz, M.A.)—Prove that the mean minimum eccentricity of an ellipse capable of resting in equilibrium on a *rough* inclined plane is $e_1 = 2(\sqrt{2} - 1)$..... 67

6259. (G. S. Carr, B.A.) — A railway carriage door of width $2a$ and mass m, with the axis through its hinges inclined α to the vertical in a plane at right angles to the rails, hangs open at rest. The carriage suddenly moves forward with an initial velocity v, and a constant acceleration afterwards f. Show

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that—(1) if the door slams, the force of the blow on the jamb is $\frac{1}{2}m[v^2+\frac{8}{3}a(f-g\sin\alpha)]^{\frac{1}{2}}$; and (2) the time of a small oscillation of the door about the position of equilibrium, is

$$4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}(f^2+g^2\sin^2\alpha)^{-\frac{1}{4}}\ldots\ldots 57$$

6266. (E. W. Symons, M.A.) — Prove that the tetrahedron of reference, and the tetrahedron formed by the tangent planes at its vertices to the conicoid

$$(l+m+n)(lyz+mzx+nxy) = mn\,xw+nl\,yw+lm\,zw,$$

are such that the lines joining corresponding vertices meet in a point; and that, when l, m, n vary, this point moves on the surface $x^{-1}+y^{-1}+z^{-1}+w^{-1}=0$. 63

6279. (Prof. Townsend, F.R.S.) — An annular lamina of uniform density, bounded internally and externally by concentric circles of any radii, being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere on the concentric sphere, whose radius is the geometric mean between those of its bounding circles; shew, by any method, that the entire resultant attraction is perpendicular to its plane 73

6281. (Prof. Wolstenholme, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a circle whose centre lies on the given asymptote. 115

6290. (Rev. F. D. Thomson, M.A.)—Find the envelop of the axis of a parabola inscribed in a given triangle 38, 100

6292 & 6325. (J. McDowell, M.A., and R. Knowles, B.A., L.C.P.)—If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1-\tfrac{1}{2}p_2+\tfrac{1}{3}p_3-\ldots+\frac{1}{n}(-\tfrac{1}{1})^{n-1}p_n = 1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}\ldots..(1),$$

$$p_1-2p_2+\ldots+(n-1)(-)^{n-2}p_{n-1}=0\ldots\ldots\ldots\ldots..(2),$$

$$\tfrac{1}{2}p_1-\tfrac{1}{3}p_2+\ldots+\frac{1}{n+1}(-)^{n-1}p_n=\frac{1}{n+1}\quad\ldots\ldots\ldots..(3). \quad 96$$

6293. (J. W. Russell, M.A.)—Two bags A and B contain each two balls, each of which is either white or black; a ball is drawn from A and put into B, and then a ball is drawn from B and put into A; in both cases the ball turns out to be white; and this double operation is repeated n times: find the probability of its not failing in the $(n+1)^{\text{th}}$ attempt. 76

6298. (A. Martin, M.A.)—If three equal circles be piled up at random on a horizontal plane, prove that the probability that the pile will stand, is $\frac{1}{16}+\frac{3\sin^{-1}\frac{1}{3}}{8\pi}-\frac{9\sqrt{15}}{256\pi}$..... 43

6307. (Prof. Crofton, F.R.S.)—The vertex A of a triangle is fixed, the base a is a given length taken anywhere along a fixed line; and from a fixed point on this line a parallel to b is drawn, intersecting c in P; find the locus of P. 24

6308 & 6283. (Prof. Francesco Tirelli.) — Se si ha un triangolo ABC, e si congiungono i vertici ad un punto O del cerchio circoscritto al triangolo, e si projetta ciascun vertice sulla congiungente il punto O agli altri due, le tre rette, che uniscono le due projezioni di ciascun vertice, concorrono in un punto della retta, che unisce le projezioni di O su lati del triangolo ABC.

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Se si ha un triangolo ABC ed un punto O del cerchio circoscritto, e si forma il quadrangolo L_1, di cui due vertici opposti sono le projezioni C′ ed A′ di O su BA, BC, mentre gli altri due sono le projezioni α_1 e γ_1 di B su OA, OC; il quadrangolo L_2 di cui due vertici opposti sono le projezioni A′ e B′ di O su CB, CA, mentre gli altri due sono le projezioni β_2 ed α_2 di C su OB, OA; ed il quadrangolo, L_3, di cui due vertici opposti sono le projezioni B′ e C′ di O su AC, AB, mentre gli altri due sono le projezioni γ e β di A su OC, OB, questi quadrangoli hanno ciascuno una coppia di lati opposti paralleli, e le coppie di lati opposti C′A′ ed $\alpha_1\gamma_1$, A′B′ e $\beta_2\alpha_2$, B′C′ e $\gamma\beta$ concorrenti in un punto. 24

6309. (Prof. Townsend, F.R.S.)—A variable conic being supposed to touch in every position the four sides of a fixed quadrilateral circumscribed to a circle; show that its two axes of figure envelope the parabola which touches the three diagonals of the quadrilateral, and has for directrix the right line which bisects their three lengths. 31

6312. (Prof. Cochez.) — Trouver le lieu géométrique du sommet d'une parabole de forme invariable qui le meut en restant constamment tangente à deux droites rectangulaires. 23

6313. (Prof. Genese, M.A.)—Prove that, with the usual notation, the distance from the centre of the inscribed circle of a triangle to the orthocentre is $(2r^2-4R^2\cos A\cos B\cos C)^{\frac{1}{2}}$ 72

6315. (Prof. Matz, M.A.)—The dome of the Capitol at Washington rises m ($=396\frac{1}{8}$ feet), above the level of the ground, and is surmounted by the Statue of Freedom, whose height is n ($=18$ feet); show that an observer whose eye is k ($=5\frac{1}{8}$ feet) above the ground, can get the best view of the Statue when his distance from the axis of the dome is

$$\{(m-h)[(m-h)+n]\}^{\frac{1}{2}} = 399{\cdot}8987372 \text{ feet}. \ldots\ldots 61$$

6319. (W. S. B. Woolhouse, F.R.A.S.)—On a given circle, of radius unity, a corner of a triangle is taken at a given distance (c) from the centre; the other two corners are taken at random on the surface; prove that the average area of all such triangles, in parts of the area of the circle, is

$$\frac{1}{\pi^2}\left(\tfrac{4}{9}+\tfrac{2}{3}c^2-\tfrac{1}{6}c^4+\tfrac{1}{36}c^6\right). \ldots\ldots 42$$

6320. (Elizabeth Blackwood.) — Given that $ax+by$ is less than e and greater than d, and that $cx+dy$ is less than a and greater than b; find the limits of x and y; a, b, c, d being all positive... 54

6321. (W. H. H. Hudson, M.A.)—A particle is placed at the extremity of the vertical minor axis of a smooth ellipse, and is just disturbed; show that, if it quit the ellipse at the end of the latus rectum, the eccentricity must satisfy the equation

$$e^6+5e^4+3e^2=5. \ldots\ldots 26$$

6326. (Rev. F. D. Thomson, M.A.) — A conic cuts a cubic in 6 points; prove that the conics of 5-pointic contact at these points meet the cubic again in 6 points on a conic. 118

6327. (Rev. T. W. Openshaw, M.A.) — Normals are drawn at the ends of a chord parallel to the tangent at the point of an ellipse whose eccentric angle is α; prove that the locus of their intersection is

$$2(ax\sin\alpha+by\cos\alpha)(ax\cos\alpha+by\sin\alpha)=(a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha \quad 27$$

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6331. (J. W. Russell, M.A.)—From the centres α, β, γ of the circles escribed to the triangle ABC, are drawn perpendiculars to the sides produced, so as to form a hexagon $\alpha\gamma'\beta\alpha'\gamma\beta'$ whose opposite sides are parallel; show that the perpendiculars from α', β', γ' on the corresponding sides of the triangle meet in a point..... 42

6335. (W. E. Wright, B.A.)—If ds be an element of an arc of an ellipse whose semi-axes are a, b; and a', b' the semi-axes of the confocal hyperbola through the element; prove that

$$\frac{ds}{da'} = \left(1 + \frac{b^2}{b'^2}\right)^{\frac{1}{2}} \text{.....} \text{.....} \quad 86$$

6337. (J. Young, B.A.)—If the equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0$$

have a common root, form the quadratic whose roots are their other roots. 29

6339. (Prof. Sylvester, F.R.S.)—Understanding by an Algebraical Perimeter of a rectilinear figure, the sum of the sides, each taken with the positive or negative sign; prove that the necessary and sufficient condition of all the curves described by points rigidly attached to the connecting link in a 3-bar linkwork being unicursal is that one of the algebraical perimeters of the quadrilateral (real or impossible) formed by the three links and the line of centres shall be equal to zero. 46

6340. (Prof. Townsend, F.R.S.)—The normal at a variable point on a fixed surface being supposed to intersect in every position a fixed right line in the space of the surface; prove, from the requisite equation of condition, that—

(a) Whatever be the position of the line in the space, the curve locus of the variable point is the intersection with the surface of another of the same order, passing through its several points of intersection with the line, and through the several points at which its tangent planes are perpendicular to the line.

(b) When the line at any of the aforesaid points of intersection is itself a normal to the surface, its point of normal intersection is a crunodal double point of the aforesaid curve, the two tangents at which intersect at right angles, and coincide in direction with the two principal tangents to the surface at the point..... 52

6341. (Prof. Minchin, M.A.)—A rigid body moves about a fixed point O, the motion of the instantaneous axis OI being completely given, as also the angular velocity of the body at each instant about it; find the components of the acceleration of any particle P, in the body along OI, a perpendicular to OI and OP, and a line perpendicular to these two directions..... 66

6342. (Prof. Genese, M.A.)—Prove that the envelop of the directrix of a parabola (1) inscribed in, or described about, a triangle, is a conic; (2) touching two straight lines and passing through a point, or passing through two points and touching a straight line, is a quartic. 119

6347. (Frederick Purser, M.A.)—A variable conic being supposed to touch four fixed right lines in a plane; show that its axes of figure envelope in general a curve of the third class, which breaks up, when the four lines touch a common circle, into the parabola of Question 6309 and the centre of the circle..... 32

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6353. (J. Venn, M.A.)—All x is either y and not z, or z and not y. All wx is either both y and z or neither of them. All xy which is not w is z. No x that is neither y nor w is z. Show that under these conditions there is no x. 51, 74

6356. (C. Leudesdorf, M.A.)—If 7 be divided by 5, given 1 and 2 over; and then successively 21 by 5, giving 4 and 1 over; 14 by 5, giving 2 and 4 over; 42 by 5, giving 8 and 2 over, and so on until the figures recur, the result of placing all the quotients after the 7 and adding a decimal point suitably is $\cdot\dot{7}14285\dot{}$, which is equal to $\frac{5}{7}$. Investigate the general form of the fractions $a : b$ which can be expressed as circulating decimals by dividing b by a in the manner explained with reference to the case of $\frac{5}{7}$. 97

6357. (J. E. A. Steggall, M.A.)—Find in how many ways a given number n may be split up into the sum of 1, 2, 3 ... different numbers. 51

6358. (R. Tucker, M.A.)—If O, P are the equidistant angle-point and the orthocentre of a triangle, and ρ_1, ρ_2, ρ_3 the radii of circles about OPA, OPB, OPC respectively; show (1) that

$$OP^2(\rho_1^{-2}+\rho_2^{-2}+\rho_3^{-2}) = 8[1-\cos(A-B)\cos(B-C)\cos(C-A)];$$

and (2) that, if any angle, as C, $= 60°$, then four of the points, as A, B, O, P, are concyclic..... 52

6362. (E. B. Elliott, M.A.)—If m lines, straight or curved, are drawn across an enclosed portion of area; prove that, if n_1 be the number of simple internal intersections of these lines with one another or themselves, n_2 the number of triple points, n_3 of quadruple, &c., formed by them, the number of separate areas into which they divide the inclosure is $1+m+n_1+2n_2+3n_3+\ldots$ 120

6365. (H. Murphy.) — If from a variable point on any side of a triangle two lines are drawn making given angles with it; prove that the circle through the points of intersection with the other sides and opposite vertex passes also through a fourth fixed point. 58

6374. (Prof. Townsend, F.R.S.)—A conic being supposed to touch the three sides of a triangle, show that the three pairs of tangents to any confocal conic from the three points of contact are three pairs of common tangents to three circles of corresponding radii having their centres at the vertices of the triangle. 103

6375. (Prof. Crofton, F.R.S.)—If a heavy particle be moving, up or down, on a curve (rough or smooth), show that, if it leaves the curve, the parabola described has contact of the second order with the curve. 49

6377. (Prof. Genese, M.A.) — A quadrilateral linkage is capable in any position of having a circle inscribed in it. If one side be fixed, find the locus of the centre of the inscribed circle..... 102

6378. (Prof. Wolstenholme, M.A.)—Prove that, if $n > 1$,

$$\int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}} = \tfrac{1}{4}\pi \left\{ \frac{\Gamma\frac{1}{2}(n-1)}{\Gamma\frac{1}{2}(n)} \right\}^2 \ldots\ldots \quad 75$$

6382. (Elizabeth Blackwood.)—If x, y, z be each taken at random between a and $-a$; find (1) the chance that $x+y+z$ will be between b and $-b$; and explain (2) any easy experimental method by which the result may be verified. [Taking a and b each 1, Miss Blackwood has found the event to happen 65 times in 100 trials.] 50

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6383. (J. Venn, M.A.)—Are there any inconsistencies or redundancies in the following rules:—(1) The *Financial* Committee shall be chosen from among the *General* Committee; (2) No one shall be a member both of the *General* and *Library* Committees unless he be also on the *Financial* Committee; (3) None of the *Library* Committee shall be on the *Financial*?..... 36

6384. (Dr. Hopkinson, F.R.S.)—A heavy wire of uniform section is carried on a series of supports in the same horizontal plane, L_r being the bending moment at the r^{th} point of support, l_r the distance between the $(r-1)^{th}$ and r^{th} support, and m the mass of the wire per unit length; prove that

$$L_{r-}l_r + 2L_r(l_r + l_{r-1}) - L_{r+1}l_{r+1} = \tfrac{1}{4}mg(l_r^3 + l_{r+1}^3)$$..... 47

6386. (W. S. McCay, M.A.)—If two circles cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar to one of the circles as the other point has to the other circle; and (2) the distance between such a pair of points subtends a right angle at an intersection of the circles. 41

6391. (J. J. Walker, M.A.)—If O, A, B, C, D are any five points in space, prove that lines drawn from the middle points of BC, CA, AB respectively parallel to the connectors of D with the middle points of OA, OB, OC, meet in one point E, such that DE passes through, and is bisected by, the centroid of the tetrahedron OABC..... 40

6392. (E. B. Elliott, M.A.)—A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2 = x_1OA_1^2 + x_2OA_2^2 + x_3OA_3^2 + x_4OA_4^2 - \Sigma(a_{12}^2 x_1 x_2)$$..... 90

6396. (C. Leudesdorf, M.A.) — Show that the square of the tangent from any point P to the circle inscribed in the triangle ABC is $\dfrac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - r(r+2R)$,

r being the radius of this circle, and R that of the circumscribed circle..... 53

6399. (Rev. H. G. Day, M.A.) — If (u, v, w) are the distances of a point from the corners A, B, C of a triangle, and (l, m, n) are constants; find the point within the triangle such that $lu + mv + nw$ shall be a minimum, when a minimum is possible. 59

6400. (J. Hammond, M.A.)—Prove that the surface

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is one of revolution, and find its axis and the equation of the generating curve (referred to its asymptotes as axes). 118

6401. (R. Knowles, B.A., L.C.P.) — Prove that the locus of the middle points of normal chords to a parabola is the curve

$$(y^2 + 2a^2)^2 - 2axy^2 + 4a^4 = 0$$..... 91

6402. (W. E. Wright, B.A.)—Find the envelop of the line joining the feet of the perpendiculars drawn on the sides of a right-angled triangle from any point in the hypotenuse. 71

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6404. (Sir James Cockle, F.R.S.)—Transform

$$y'' + 2\,(\mathrm{A}\cot x + \mathrm{E}\tan x)\,y' + \left\{\mathrm{Æ}^2 - (\mathrm{A}-\mathrm{E})^2\right\} y = 0 \ldots (1)$$

into an equation of the form called by Boole binomial..... 45

6407. (Prof. Crofton, F.R.S.)—If, in a triangle, $\mathrm{C} = 60°$, prove that $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$..... 67

6410. (Prof. Genese, M.A.)—Conics are drawn with a common focus, and constant major axis: if their centres lie on a fixed circle, prove that the envelop of the series will be a conic having the same focus..... 48

6414. (Elizabeth Blackwood.)—If x, y, z be each taken at random between 1 and -1, find the chances (1) that $x+y$, $y+z$, $z+x$ are all three positive, (2) that $x+y-xy$, $y+z-yz$, $z+x-zx$ are all three positive..... 116

6415. (C. W. Merrifield, F.R.S.)—A regular skeleton dodecahedron is made of 30 equal wires connected 3 by 3 with universal joints. It is then flattened symmetrically, so that two opposite pentagonal faces retain their shape—these two faces being connected with the corners of a regular decagon by 10 of the edges. Show that, in order to make the construction possible, each pentagon much be turned through $\frac{1}{2}\tan^{-1}2$. 94

6416. (Dr. Hopkinson, F.R.S.)—An endless slightly extensible strap is stretched over two equal pulleys; prove that the maximum couple which the strap can exert on either pulley is

$$\frac{2a\,(c+\pi a)}{c\coth\frac{1}{2}\mu\pi + \frac{2a}{\mu}}\,\mathrm{T},$$

where a is the radius of either pulley, c the distance of their centres, μ the coefficient of friction, and T the tension with which the strap is put on..... 67

6417. (C. J. Monro, M.A.)—The birds carry away the seed as fast as it falls, but the sower has the start of them by n grains. What is the probability that, after they have carried away x grains, y of those left are of the original n? (See *Origin of Species*, 5th ed., p. 381.)..... 84

6420. (Donald McAlister, D.Sc.)—A regular octohedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant..... 70

6422. (E. B. Elliott, M.A.)—Prove that the envelop of the line

$$x\cos\phi + y\sin\phi = ae^{\phi} + be^{-\phi},$$

where ϕ is a variable parameter, is the evolute of the evolute of the evolute of its own evolute..... 104

6429. (R. Tucker, M.A.)—Given two tangents to a parabola, and the point on the axis through which the chord of contact passes; construct the parabola. 97

6433. (J. W. McKenzie, B.A.) — Find, if possible, a value of n (other than $n = 24$), so that the sum of the squares of the first n natural numbers may be a square. 106

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6435. (W. E. Wright, B.A.)—Eliminate a between the equations

$$(x^2\cos\beta+\cos a)\cos a+(x+\sin a)\sin a+x\sin\beta=0,$$

$$(x^2\cos a+\cos\gamma)\cos\gamma+(x+\sin\gamma)\sin\gamma+x\sin a=0 \ldots\ldots 78$$

6438. (Prof. Cayley, F.R.S.)—Find, at any point of a plane curve, the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point..... 78

6439. (Prof. Crofton, F.R.S.)—Prove that

$$\int e^{kx^2}\,dx = e^{kx^2}(x^{-1}+1\,.\,x^{-3}+1\,.\,3x^{-5}+1\,.\,3\,.\,5x^{-7}+\ldots)\ldots\ldots 80$$

6441. (The late Prof. Clifford, F.R.S.)—It is known that, if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that, if five lines be given, the five points so belonging to their five tetragrams lie on a circle [MIQUEL'S Theorem, see *Diary* for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle. 80

6442. (Prof. Minchin, M.A.)—A body of any shape, with a plane base, rests with this base on a rough horizontal plane; a heavy beam movable round a horizontal axis fixed in the plane rests at a single point against the body, the vertical plane through the beam containing the centre of gravity of the body. Show that limiting equilibrium of the system is impossible unless the normal to the surface of contact of the beam and body makes with the vertical an angle greater than the sum of the angles of friction between the body and the beam and the body and the ground..... 83

6443. (Prof. Tait, M.A.)—Show that, whatever functions of x be represented by y and z, we have always $\dfrac{\int yz\,dx}{\int y\,dx} > e^{\frac{\int y\log z\,dx}{\int y\,dx}}$, all the integrals being taken between the same limits of x; and all the quantities involved being positive. 86

6444. (Prof. Matz, M.A.)—Three points taken at random, one on each side of a plane triangle, are joined by straight lines; show (1) that the mean area of the triangle thus formed is one-fourth of the area of the triangle; and (2) generalize the problem for any number of points taken in different areas or volumes..... 87

6447. (Christine Ladd.)—The width of a croquet hoop, the thickness of its wires, and the diameter of a ball are given: the ball being in a given position, show how to find the conditions that it may just be possible for it to go through the hoop, (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection... 81

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6454. (W. H. H. Hudson, M.A.)—If NQ, CY be the perpendiculars from N, the intersection of the ordinate and directrix, and from C, the vertex, on the tangent at any point P of a catenary, and O be the point where the directrix meets the axis, prove that $QY : OC = \widehat{CP} - ON : PN$. 89

6457. (E. B. Elliott, M.A.)—P is one of the intersections of a lemniscate with the circle on the intercept SS′ between its foci as diameter. Prove that the two curves cut at an angle of 60°, and that the bisector of the angle SPS′ is also the bisector of the angle between the two tangents at P. 120

6460. (J. W. Russell, M.A.) — Show that, as the infinite branch of the curve $r = \frac{2a}{1-\theta^2}$ rolls on its initial tangent, the pole will describe a semicircle 89

6461. (H. McColl, B.A.) — Show that the weakest addition that must be made to the premises A : a, B : b, C : c, &c., to justify the inference Q : q, is the implication $Qq' : Aa' + Bb' + Cc' + \ldots$ 85

6463. (H. L. Orchard, M.A.)—Find the locus of the intersection of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of a chord through the point $(a, 0)$. 105

6468. (A. Martin, M.A.)—An auger hole is made through the centre of a sphere; show that the chance that the volume removed does not exceed one-n^{th} of the sphere is $\left[1-\left(1-\frac{1}{n}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}}$. 119

6479. (Prof. Wolstenholme, M.A.)—Prove that the asymptotic cone of a quadric $u = 0$ is $A\left(\frac{\delta u}{\delta x}\right)^2 + 2F\frac{\delta u}{\delta y}\frac{\delta u}{\delta z} + \&c. = 0$, and the equations of its axes are

$$\frac{A\frac{\delta u}{\delta x} + H\frac{\delta u}{\delta y} + G\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta x}} = \frac{H\frac{\delta u}{\delta x} + B\frac{\delta u}{\delta y} + F\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta y}} = \frac{G\frac{\delta u}{\delta x} + F\frac{\delta u}{\delta y} + C\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta z}},$$

where $A = \frac{\delta^2 u}{\delta y^2}\frac{\delta^2 u}{\delta z^2} - \left(\frac{\delta^2 u}{\delta y\,\delta z}\right)^2$, $F = \frac{\delta^2 u}{\delta z\,\delta x}\frac{\delta^2 u}{\delta x\,\delta y} - \frac{\delta^2 u}{\delta y\,\delta z}\frac{\delta^2 u}{\delta x^2}$, &c. 82

6480. (Prof. Genese, M.A.) — Given a tangent, a point, and a focus of a conic; prove that the locus of the centre is a conic... 112

6483. (R. Tucker, M.A.)—From P, any point on the line $x = 2a$, tangents PT, PT′ are drawn to a parabola; the normals at T, T′ (as is known) intersect on the curve (in Q). Prove that TT′ passes through a fixed point on the axis, on which axis also lies the centroid of Q, T, T′, the circle about PTQT′ passes through the vertex, and its envelop is a cubic curve; and the orthocentre of QTT′ lies on the diameter through P and on a parabola. 115

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6484. (W. H. Besant, M.A.)—Having given two circles of radii R and r, and that the distance between their centres is $(R^2-2Rr^{\frac{1}{2}})^{\frac{1}{2}}$, prove that an infinite number of triangles can be drawn which shall be inscribed in one circle and circumscribed about the other..... 117

6488. (W. B. Grove, B.A.)—If my son is to enter either the law or the church, he must go either to Oxford or to Cambridge. If he goes to Oxford without entering the law, or to Cambridge without entering the church, he will get a legacy at his uncle's death. He will fail to get the legacy under the following circumstances:—If he will not go to Oxford, and at the same time will not enter the church; or if he will not go to Cambridge, and at the same time will not enter the law—and under *no other* circumstances. I decide that he must enter either the law or the church: will he or will he not get the legacy?..... 80

6489. (T. P. Kirkman, M.A., F.R.S.)—On a square chequer of n^2 small squares, stand n^2-1 counters marked 1, 2, 3 ... (n^2-1), in any order S, covering all the squares except the last, at the right-hand corner; this last is occupied by a King, visible or invisible. The King, on whatever square he happens to stand, may exchange places only with any counter on a compartment collateral with his. The problem for solution is, to move the King about the board so that, when he has returned to his own final square, the order S shall be reduced to the natural order, the counters marked a, b, c ... standing on the $a^{\text{th}}, b^{\text{th}}, c^{\text{th}}$ ... squares. Required a rule whereby, on inspection of the order S, the solution can be proved possible or impossible..... 113

6499. (J. L. McKenzie, B.A.)—If x_n and y_n be the n^{th} positive integral solution of $x^2-Ny^2=1$, prove that $x_{n+p}=2x_px_n-x_{n-p}$, and $y_{n+p}=2x_py_n-y_{n-p}$. 114

6503. (Sir James Cockle, F.R.S.)—Transform the equation (1) of Question 6404 into

$$Y''+\left(\varpi\tan t-\frac{i\lambda}{\varpi}\right)Y'+\tfrac{1}{4}\left\{1-\left(\varpi+\frac{\lambda}{\varpi}\right)^2-4N\right\}Y=0\ldots(\text{i.}),$$

where $i=(-1)^{\frac{1}{2}}$; show that (i.) admits of a continuous transformation whereby ϖ is changed into $\varpi-2$, $\varpi-4$, and so on; and notice the case of $\varpi=0$..... 109

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6243. (By Prof. SYLVESTER, F.R.S.)—If $2f^2+g^2$ is a prime and f an odd number, prove that $fy^2+2gxy-2fx^2 = \pm 1$ is soluble in integers.

I. *Solution by* SAMUEL ROBERTS, M.A.

In this case $2f^2+g^2$ is a prime of the form $8m+3$.

Now, if $P^{\frac{1}{2}} = (2f^2+g^2)^{\frac{1}{2}}$ be developed as a continued fraction in the usual way, and $$\frac{P^{\frac{1}{2}}+I}{D}, \quad \frac{P^{\frac{1}{2}}+I_1}{D_1}, \quad \frac{P^{\frac{1}{2}}+I_2}{D_2}$$ are three complete successive quotients in the process, GOEPEL has shown that (1), if $f<2g$, we shall arrive at

$$D=f, \quad I_1=g, \quad D_1=2f, \quad \text{or} \quad D=2f, \quad I_1=g, \quad D_1=f;$$

or (2), if $f>2g$, we shall arrive at

$$D=f+2g, \quad I_1=f-g, \quad D_1=f, \quad D_2=f-2g, \quad I_2=f+g.$$

In case (1), the forms $(\pm f, g, \mp 2f)$, $(\pm 2f, g \mp f)$, are reduced forms of the principal class. In case (2), the forms

$$[\pm(f+2g), f-g, \mp f], \quad (\pm f, f+g), \quad \mp(f-2g)$$

are reduced forms of the principal class.

But $(f+2g, f-g, -f)$ is properly transformed into $(2f, -g, -f)$ by the substitution $\begin{pmatrix} 1, & 0 \\ 1, & 1 \end{pmatrix}$, and $[f, f+g, -(f-2g)]$ is properly transformed into $(f, g, -2f)$ by the substitution $\begin{pmatrix} 1, & -1 \\ 0, & 1 \end{pmatrix}$. Hence, in both cases, we can make $(f, g, -2f) = 1$, $(2f, -g, -f) = 1$, in integers.

II. *Solution by the* PROPOSER.

Since $x^2-Ay^2 = 1$ is always soluble in integers, we may make

$$2f^2+g^2 = A, \quad u^2-Av^2 = 1,$$

and write $$u+1 = \sigma p^2, \quad u+1 = A\sigma q^2$$

(p, q denoting relative primes).

Hence $$p^2 - Aq^2 = \frac{2}{\sigma} = \pm 1 \text{ or } \pm 2.$$

In fact we must have $$p^2 - Aq^2 = 1 \text{ or } -2,$$

because A being odd, and of the form $2f^2 + g^2$ (where f is odd), is of the form $8n + 3$; and consequently the equations

$$p^2 + 1 = Aq^2 \quad \text{and} \quad p^2 - 2 = Aq^2$$

are each impossible, as is apparent, without recourse to the theory of quadratic residues, by considering each of them with regard to the modulus 8.

If $$p^2 - Aq^2 = 1, \quad v = 2pq,$$

and substituting p, q for u, v, and p_1, q_1 for p, q, we shall obtain in like manner $$p_1^2 - Aq_1^2 = 1 \text{ or } = -2.$$

If $p_1^2 - Aq_1^2 = 1$, we must have $v = 2pq = 4pp_1q$, and continuing this process, since v can only contain 2 a finite number of times, we must come at last to a cessation of the alternative, and to the equation

$$\pi^2 - A\phi^2 = -2, \quad i.e., \quad \pi^2 + 2A = \phi^2,$$

where π, ϕ will both be odd.

Every prime divisor then of $\pi^2 + 2$ we know is of the form $8n + 1$, or $8n + 3$, and therefore is itself of the form $r^2 + 2s^2$, whence, by a known principle of the theory of numbers, it follows that we must have

$$\pi \pm \sqrt{-2} = (g + f\sqrt{-2})(g + x\sqrt{-2})^2.$$

Whence, equating the imaginary parts, we find

$$fy^2 + 2gxy - fy^2 = \pm 1.$$

6005. (By the Editor.)—If three lengths are taken at random, and the limits be the same (and any magnitude) for all three, show that the respective probabilities of being able to form with them (1) a triangle of any kind, (2) an obtuse-angled triangle, are $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}\pi$.

Solution by C. J. Monro, M.A.

1. It is meant that dx, dy, dz shall be proportional to the chance that the lengths are between x, y, z and $x + dx$, $y + dy$, $z + dz$ respectively. Therefore the chance required is the range of a point having the lengths for rectangular coordinates, as a fraction of a cube with a corner at the origin and three edges along the axes, of a length equal to the common limit. The well-known condition excludes the point from the three tetrahedrons cut off by planes through the origin and adjacent face-diagonals. Having common edges in the faces of the cube, these three have no volume in common; and the volume of each is $\frac{1}{6}$.

2. Cut off quarter-cones having their axes along the coordinate axes and generators along face-diagonals, and you leave $1 - \frac{1}{4}\pi$ for the chance that the triangle is acute-angled, and therefore $\frac{1}{4}\pi$ for the chance of an obtuse-angled triangle. [Another solution of the first part is given, under Quest. 1977, on p. 45 of Vol. IX. of the *Reprint*.]

Note on Solution of Linear Differential Equations with Constant Coefficients. *By* Prof. Wolstenholme, M.A.

In any equation $f\left(\frac{d}{dx}\right) y = X$, where f is a rational integral function, and X a function of x, if one term in X be $\cos nx$ or $\sin nx$, the corresponding term in y is most easily found as follows:—Denoting $\frac{d}{dx}$ by D, let

$$f(D) = \phi_1(D^2) + D\phi_2(D^2), \text{ then } \frac{1}{f(D)} = \frac{\phi_1(D^2) - D\phi_2(D^2)}{(\phi_1 D^2)^2 - D^2(\phi_2 D^2)^2};$$

and therefore $$\frac{1}{f(D)} \cos nx = \frac{\phi_1(-n^2)\cos nx + n\phi_2(-n^2)\sin nx}{[\phi_1(-n^2)]^2 + n^2[\phi_2(-n^2)]^2},$$

and $$\frac{1}{f(D)} \sin nx = \frac{\phi_1(-n^2)\sin nx - n\phi_2(-n^2)\cos nx}{[\phi_1(-n^2)]^2 + n^2[\phi_2(-n^2)]^2}.$$

This will not apply when $f(-n^2) = 0$, but it will then much simplify the work to write $f(D) = (D^2 + n^2)^r \phi(D)$, and to operate first with $\frac{1}{\phi(D)}$ in the way just shown, then

$$\frac{1}{(D^2+n^2)^r} \cos nx = \tfrac{1}{2} \frac{x^r}{r!} \left(\frac{e^{nix}}{(2ni)^r} + \frac{e^{-nix}}{(-2ni)^r} \right),$$

and similarly for $\sin nx$.

6312. (By Prof. Cochez.)—Trouver le lieu géométrique du sommet d'une parabole de forme invariable qui le meut en restant constamment tangente à deux droites rectangulaires.

Solution by Prof. Townsend, F.R.S.; Prof. Tanner, M.A.; *and others.*

If, in any position of the parabola, x and y be the coordinates of the vertex to the two rectangular lines as axes, m the modulus of the curve, and θ the angle made by its axis of figure with the axis of x; then since, from the geometry of the parabola, we have $x \cos\theta = m \sin^2\theta$ and $y \sin\theta = m \cos^2\theta$, therefore at once, by elimination of θ, we get

$$x^{\frac{1}{3}} y^{\frac{1}{3}} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = m^2;$$

which accordingly is the equation, in its most symmetrical form, of the locus in question.

Cleared of radicals, this equation is manifestly equivalent to

$$x^2y^2(x^2 + y^2 + 3m^2) = m^6;$$

from which it appears at once that the locus is of the sixth degree, passes through the two circular points at infinity, and consists of four similar and equal hyperbolical branches, lying asymptotically in the four right angles determined by the axes, touching externally at their four vertices the circle of radius m round the origin as centre, and having all inflexional contact with the axes at their two real intersections with infinity.

6307. (By Prof. Crofton, F.R.S.)—The vertex A of a triangle is fixed, the base a is a given length taken anywhere along a fixed line; and from a fixed point on this line a parallel to b is drawn, intersecting c in P; find the locus of P.

Solution by C. J. Monro, M.A.; J. E. A. Steggall, B.A.; *and others.*

A′ being the fixed point in BC, complete the parallelograms ABA′C′, ACA′B′: if C may be on either side of B, the locus will be symmetrically related to the fixed points and lines. Join AA′, and draw PM an ordinate to it parallel to the other fixed lines. Then

$$\frac{AM}{AA'} = \frac{AP}{AB} = \frac{CA'}{CB} = \frac{AA'}{MA'} \cdot \frac{MP}{CB'}, \text{ or } AM \,.\, MA' = \frac{AA'^2}{a} \,.\, MP.$$

Thus P lies on either of the parabolas which cut the lines at A, A′, and cut the parallel through the mid-point of AA′ at a distance $\frac{1}{4}a$ from that point. In the parallelogram APA′P′, P and P′ belong to different parabolas.

6241. (By E. W. Symons, B.A.)—The roots of the equation

$$x^3 + \lambda x = (x^2 + 1)(2\lambda + \mu)^{\frac{1}{2}}$$

are all real and positive; prove that neither λ nor μ can be less than 9.

Solution by R. Tucker, M.A.; J. A. Kealy, M.A.; *and others.*

Writing the equation in the form $x^3 + ax^2 + \lambda x + a = 0$, we have, in the case supposed, (see Todhunter's *Theory of Equations*, Art. 173,)

$$4a^4 + a^2(27 - 18\lambda - \lambda^2) + 4\lambda^3 = \text{negative quantity} = U \ldots\ldots\ldots(1).$$

Now, $16U = \{8a^2 - (\lambda^2 + 18\lambda - 27) + [(\lambda - 1)(\lambda - 9)^3]^{\frac{1}{2}}\}$

$$\times \{8a^2 - (\lambda^2 + 18\lambda - 27) - [(\lambda - 1)(\lambda - 9)^3]^{\frac{1}{2}}\}.$$

Hence, that (1) may be satisfied, ($\because \lambda$ must be + ve and > 1) $\lambda \not< 9$, and $8a^2$, *i.e.*, $8(2\lambda + \mu) \not< (81 + 162 - 27) \not< 216$, *i.e.*, $2\lambda + \mu \not< 27$, or $\mu \not< 9$.

6308, 6283. (By Prof. Francesco Tirelli.)—Se si ha un triangolo ABC, e si congiungono i vertici ad un punto O del cerchio circoscritto al triangolo, e si projetta ciascun vertice sulla congiungente il punto O agli altri due, le tre rette, che uniscono le due projezioni di ciascun vertice, concorrono in un punto della retta, che unisce le projezioni di O su lati del triangolo ABC.

Se si ha un triangolo ABC ed un punto O del cerchio circoscritto, e si forma il quadrangolo L_1, di cui due vertici opposti sono le projezioni C′ ed

A′ di O su BA, BC, mentre gli altri due sono le projezioni a_1 e γ_1 di B su OA, OC; il quadrangolo L_2 di cui due vertici opposti sono le projezioni A′ e B′ di O su CB, CA, mentre gli altri due sono le projezioni β_2 ed a_2 di C su OB, OA; ed il quadrangolo L_3, di cui due vertici opposti sono le projezioni B′ e C′ di O su AC, AB, mentre gli altri due sono le projezioni γ e β di A su OC, OB, questi quadrangoli hanno ciascuno una coppia di lati opposti paralleli, e le coppie di lati opposti C′A′ ed $a_1\gamma_1$, A′B′ e $\beta_2 a_2$, B′C′ e $\gamma\beta$ concorrenti in un punto.

Solution by D. Edwardes; Professor Genese, M.A.; *and others.*

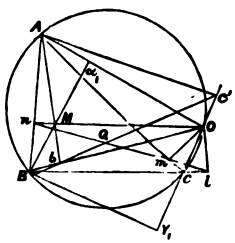
(6308.) Let bc' be the pedal line of A with regard to OBC, and lmn that of O with regard to ABC; and let bc', ln meet in Q. Then

$\angle Qnc' = \angle mAc'$ (in circle $AnmOc'$)

$= \frac{1}{2}\pi - \angle ACc' = \frac{1}{2}\pi - \angle ACO = \frac{1}{2}\pi - \angle ABO$;

and $\angle Qc'n = \angle bOn = \frac{1}{2}\pi - \angle ABO$,

therefore $\angle bQn = \pi - 2\angle ABO$.



Therefore Q lies on the nine-point circle of ABO, and Q is easily seen to lie on the nine-point circle of ACO. Hence, since the pedal lines A (OBC), O (ABC, intersect on the nine-point circles of ABO and ACO, it follows that the pedal lines A (OBC), B (OAC) intersect on the nine-point circles of ABO and ABC. Hence these four pedal lines meet in a point, through which pass the nine-point circles of the four triangles. [It can be shown that the pedal line O ABC) bisects the line joining O to the orthocentre of ABC.—(W. H. Besant, in Math. Society's *Proceedings*.) Hence, if H, K, L, M be the orthocentres of ABC, BCD, COA, OAB, then HO, KA, LB, MC meet in a point.]

This also proves the latter part of Question 6283.

For the first part, we have $\angle CB\gamma_1 = \frac{1}{2}\pi - \angle OCl = \frac{1}{2}\pi - \angle A = \angle AOn$. Also $\angle ln\gamma_1 = \angle C\beta\gamma_1$ (in same segment), and $\angle nla_1 = \angle a_1On$ (in same segment); therefore $\angle ln\gamma_1 = \angle nla_1$, therefore $n\gamma_1$ is parallel to la_1, and similarly for the other quadrilaterals.

6232. (By J. J. Walker, M.A.)—A system of forces being reduced to two, one of which passes through a fixed point and makes a given angle with the resultant of translation; show that the other has for its envelop an ellipse in the plane through the fixed point perpendicular to the axis of principal moment for that point; and that the ellipse becomes a circle when the fixed point is Poinsot's centre.

Solution by W. J. C. Sharp, M.A.; G. H. Hopkins, M.A.; *and others.*

Take OG, the axis of principal moment at the given point O, as axis of z, and the plane through the resultant R as plane of xz; so that $\sin\psi$, 0, $\cos\psi$, are the direction cosines of R; and let P in the direction α, β, γ, and

Q in that of α', β', γ', acting at $(x'y'z')$, be the two forces, and A the given angle between P and R. Then we have

$$\cos\alpha\sin\psi + \cos\gamma\cos\psi = \cos A,\quad R\sin\psi = P\cos\alpha + Q\cos\alpha' \ldots(1, 2),$$

$$R\cos\psi = P\cos\gamma + Q\cos\gamma',\quad 0 = P\cos\beta + Q\cos\beta' \ldots\ldots(3, 4),$$

$$0 = Q(y'\cos\gamma' - z'\cos\beta'),\quad 0 = Q(z'\cos\alpha' - x'\cos\gamma') \ldots\ldots(5, 6),$$

$$G = Q(x'\cos\beta' - y'\cos\alpha') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (5) and (6), $\frac{x'}{\cos\alpha'} = \frac{y'}{\cos\beta'} = \frac{z'}{\cos\gamma'}$, which imply a single resultant, or $z' = 0$, $\cos\gamma' = 0$; *i.e.*, Q lies in the plane of xy, and the equation to its line of action is $x\cos\beta' - y\cos\alpha' = \frac{G}{Q}$. Now, from (1), (2), and (3),

$$R = P\cos A + Q\cos\alpha'\sin\psi,$$

and from (2), (3), and (4), $R^2 - 2RQ\cos\alpha'\sin\psi + Q^2 = P^2$.

Therefore $Q^2(1 - \cos^2\alpha'\sin^2\psi) = P^2\sin^2 A$; therefore the equation to Q is $y + \frac{G\sin\psi}{R} = x\tan\alpha' - \frac{G\cot\alpha}{R}(\tan^2\alpha' + \cos^2\psi)^{\frac{1}{2}}$, the envelop of which is the ellipse $x^2 + \left(y + \frac{G\sin\psi}{R}\right)^2 \frac{1}{\cos^2\psi} = \frac{G^2}{P^2\sin^2 A}$, which becomes a circle when $\psi = 0$, that is to say, when the point is Poinsot's centre.

6321. (W. H. H. Hudson, M.A.)—A particle is placed at the extremity of the vertical minor axis of a smooth ellipse, and is just disturbed; show that, if it quit the ellipse at the end of the latus rectum, the eccentricity must satisfy the equation $e^6 + 5e^4 + 3e^2 = 5$.

Solution by Prof. Scott, M.A.; D. Edwardes; *and others.*

If v be the velocity at the one end of the minor axis, $v^2 = 2g\left(b - \frac{b^2}{a^2}\right)$; and the chord of curvature at the extremity of the *latus rectum*, parallel to the minor axis, is $2a(1 - e^4)$; therefore

$$v^2 = 2g \cdot \tfrac{1}{4} \cdot 2a(1 - e^4);\ \text{hence}\ 2\left(b - \frac{b^2}{a}\right) = a(1 - e^4),$$

or $$2 = (1 - e^2)^{\frac{1}{2}}(3 + e^2),\ \text{whence}\ e^6 + 5e^4 + 3e^2 = 5.$$

6000. (By W. S. B. Woolhouse, F.R.A.S.)—The separate probabilities of three partially dependent events are given as well as the probabilities of the joint occurrence of every two events and also the joint occurrence of all three. In a proposed case, however, it is known that two of the

events have failed, but it is unknown which; determine the probability of the failure also of the remaining event.

Solution by C. J. MONRO, M.A.

In BOOLE's forms let x, y, z be the events, and let a, b, c, l, m, n, t be the chances of x, y, z, yz, zx, xy, xyz.

These given, so are the relative frequencies of the known circumstance

$$x(1-y)(1-z)+y(1-x)(1-z)+z(1-x)(1-y)+(1-x)(1-y)(1-z) = 1-yz-zx-xy+2xyz,$$

and of the circumstance in question $(1-x)(1-y)(1-z)$. Therefore the chances of this is given as a ratio between the two; namely,

$$\frac{1-a-b-c+l+m+n-t}{1-l-m-n+2t}.$$

6327. (By the Rev. T. W. OPENSHAW, M.A.)—Normals are drawn at the ends of a chord parallel to the tangent at the point of an ellipse whose eccentric angle is α; prove that the locus of their intersection is

$$2(ax\sin\alpha+by\cos\alpha)(ax\cos\alpha+by\sin\alpha) = (a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha.$$

Solution by J. E. A. STEGGALL, B.A.; R. L. KNOWLES, L.C.P.; *and others.*

The eccentric angles of the other points are $(\alpha+\theta, \alpha-\theta)$; hence the normals are

$$\frac{ax}{\cos(\alpha\pm\theta)} - \frac{by}{\sin(\alpha\pm\theta)} = a^2-b^2;$$

therefore

$$(ax\sin\alpha-by\cos\alpha)\cos\theta = (a^2-b^2)\sin 2\alpha\cos 2\theta \quad \ldots\ldots(1),$$

$$(ax\cos\alpha+by\sin\alpha)\sin\theta = (a^2-b^2)\cos 2\alpha\sin 2\theta \quad \ldots\ldots(2).$$

Multiply by $\cos 2\alpha\sin\theta$ and $\sin 2\alpha\cos\theta$, and subtract,

$$\cos\theta\sin\theta\,(ax\sin\alpha+by\cos\alpha) \quad \ldots\ldots(3).$$

Multiply (2) and (3), and we get the result in the Question.

6019, 6088. (By T. R. TERRY, M.A.)—If $a<1$ and m and n are zero, or any positive integers, and

$$f(n,m) = \frac{(2n)!}{2^{2n}n!\,n!}\left\{1-\frac{n-m}{n+1}ma^2+\frac{(n-m)(n-m+1)}{n+1\,.\,n+2}\cdot\frac{m(m-1)}{1\,.\,2}a^4-\ldots\right\},$$

prove that

$$4a^2f(n,m) = -f(n-1,m+1)+2(1+a^2)f(n-1,m) - (1-a^2)^2f(n-1,m-1) \quad \ldots\ldots(1),$$

$$\frac{df(n,m)}{da} = 4a\frac{(n-m)(n-m+1)}{2n+1}f(n+1,\ m-1) - 2a(n-m)f(n,m-1)\ldots(2),$$

$$f(n,m+1) = (1+a^2)f(n,m) - \frac{4a(n-m)}{2n+1}f(n+1,m)\ \ldots\ldots\ldots(3).$$

Solution by the PROPOSER; J. HAMMOND, M.A.; *and others.*

It may be shown that, if $\Delta = 1 - 2a\cos x + a^2$,

$$\int_0^\pi \frac{\sin^{2n}x\,dx}{\Delta^{n-m}} = \pi f(n,m)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Hence (1) follows from the obvious truth that

$$\frac{\sin^2 x}{\Delta^2} = -\frac{1}{4a^2} + \frac{1+a^2}{2a^2}\frac{1}{\Delta} - \frac{(1-a^2)^2}{4a^2}\frac{1}{\Delta^2}.$$

Differentiating (a) with respect to a, we get (2); and, integrating $\int_0^\pi \frac{\sin^{2n}x\cos x\,dx}{\Delta^{n-m}}$ by parts, we get (3).

[The restriction on m is unnecessary. The results may also be obtained from the known properties of the hypergeometric series. See also "Notes on a Class of Definite Integrals," by the Proposer, in Vol. XI. of the *Proceedings* of the Mathematical Society.]

5973 & 6002. (By J. L. MACKENZIE, B.A.)—Solve the functional equation $\phi(ax+\beta) = \gamma\phi(x) + u$, where α, β, γ are constants and u is a known function of x; and, as particular cases, solve the equations

$$\phi(a-x)\phi(x) + b[\phi(a-x) + \phi(x)] = c^2, \quad \phi(mx) = \phi(x) + ax\log x.$$

Solution by J. HAMMOND, M.A.; H. STABENOW, M.A.; *and others.*

Let P denote an operator such that $P\phi(x) = \phi(ax+\beta)$; C a const
of solution, that is, a function unaffected by the operation P; and
particular solution of the equation $P = \gamma$, that is, a function
$PS = \gamma S$; then the solution of the general form is
$\phi(x) = CS + (P-\gamma)^{-1}u$, for $P(CS) = PC \cdot PS = \gamma CS$, and

(Qu. 5973.) Adding b^2 to each side and taking logar

$$\log[\phi(a-x)+b] + \log[\phi(x)+b] = \log(b^
$$

or $(P+1)\log[\phi(x)+b] = \log(b^2+c^2)$ $[a=-1, \beta$

And the solution is $\log[\phi(x)+b] = CS + \frac{1}{P+1}\log($

$$= CS + \tfrac{1}{2}\log(b^2+c^2)\ [\sin$$

Here C, S are arbitrary functions $F(x)$, $f(x)$,

$$F(x) = F(a-x),\ f(x$$

and we may without loss of gene
$S = \frac{1}{2}a - x$, C being in this case any

When $S = \frac{1}{2}a - x$ and $C = -a^{-1}\log(b^2+c^2)$, we have

$$\log[\phi(x)+b] = \frac{x}{a}\log(b^2+c^2), \text{ or } \phi(x)+b = (b^2+c^2)^{\frac{x}{a}}.$$

When $S = \log\tan\frac{\pi x}{2a}$, and $C = 1$, $\phi(x)+b = (b^2+c^2)^{\frac{1}{2}}\tan\frac{\pi x}{2a}$.

Both of these cases are included in the general solution

$$\phi(x) = (b^2+c^2)^{\frac{1}{2}}e^{CS} - b.$$

(Qu. 6002.) Here $\phi(x) = C + \frac{1}{P-1}ax\log x \quad [\alpha = m, \beta = 0, \gamma = 1]$.

Now $P^r ax\log x = am^r x\log(m^r x) = am^r x\log x + a\mu m^r x\log m$.

Thus $F(P)\,ax\log x = ax\log x\,F(m) + am\,x\log m\,F'(m)$,

and the solution becomes

$$\phi(x) = C + \frac{ax\log x}{m-1} - \frac{amx\log m}{(m-1)^2},$$

where C is unaltered by the change of x into mx.

6337. (By J. Young, B.A.)—If the equations

$$ax^2+bx+c = 0, \quad a_1x^2+b_1x+c_1 = 0$$

have a common root, form the quadratic whose roots are their other roots.

Solution by G. Heppel, M.A.; Prof. Genese, M.A.; *and others.*

Let $(-\alpha, -\beta)$ be the roots of the first, $(-\alpha, -\gamma)$ of the second; then

$$\alpha+\beta = \frac{b}{a}, \quad \alpha\beta = \frac{c}{a}, \quad \alpha+\gamma = \frac{b_1}{a_1}, \quad \alpha\gamma = \frac{c_1}{a_1};$$

whence $$\alpha = \frac{ac_1-a_1c}{ab_1-a_1b} = \frac{bc_1-b_1c}{ac_1-a_1c},$$

therefore $$(ac_1-a_1c)^2 = (ab_1-a_1b)(bc_1-b_1c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Since $\beta = \frac{ab_1-a_1b}{ac_1-a_1c}\cdot\frac{c}{a}$, $\gamma = \frac{ab_1-a_1b}{ac_1-a_1}\cdot\frac{c_1}{a_1}$, the required equation is

$$x^2 + \frac{ab_1-a_1b}{ac_1-a_1c}\left(\frac{c}{a}+\frac{c_1}{a_1}\right)x + \left(\frac{ab_1-a_1b}{ac_1-a_1c}\right)^2\cdot\frac{cc_1}{aa_1},$$

or, by (A), $aa_1(bc_1-b_1c)x^2 + (a^2c_1^2-a_1^2c^2)x + cc_1(ab_1-a_1b) = 0$.

6205. (By W. J. C. Sharp, M.A.) — Prove that, according as the polar conic of a point on a cubic is an hyperbola, a parabola, or an ellipse, two lines, one line, or no line, can be drawn through the point, so as to be terminated by the curve and bisected at the point.

Solution by J. J. WALKER, M.A.

According to my method of treating the intersections of a transversal, $lx + my + nz = 0$, with the curve $u = 0$ [*Proceedings of Math. Soc.*, Vol. IX., p. 227 (4)] the condition for the bisection of the chord $lx \ldots$ at the point (xyz) on u is (1) $a\lambda^2 + \ldots + 2h\lambda\mu = 0$, where $\lambda = m \sin C - n \sin B, \ldots$, and consequently (2) $\lambda \sin A + \mu \sin B + \nu \sin C \equiv 0$, $a \ldots h$ standing for the second differential coefficients of u, so that the polar conic is $ax^2 + \ldots 2hxy = 0$. Eliminating λ between (1) and (2), the discriminant of the equation for the two values of $\mu : \nu$ is, as is well known, $(-bc + f^2) \sin^2 A + \ldots$, which is negative, zero, or positive, as the polar conic is a hyperbola, parabola, or ellipse.

5791. (By W. H. H. HUDSON, M.A.)—Let AB, BC, CD be equal arcs of a circle, TA, TD be the tangents at A, D; AB, DC meet in Y; XB, XC, the tangents at B, C, meet AD produced in E, F: show that, when B, C, D, move up to A, the ultimate ratio of the triangles TAD, XEF, YAD, are as 27 . 25 : 18.

Solution by the Rev. J. L. KITCHIN, M.A.; NARENDRA LAL; *and others.*

Let AOK (= 2) be a diameter of the circle; and let AB, BC, CD subtend at the centre O an angle θ; then $\Delta YAD = \sin\theta \sin^2 \frac{3}{2}\theta$,

$$\Delta TAD = \sin^3 \tfrac{3}{2}\theta,$$

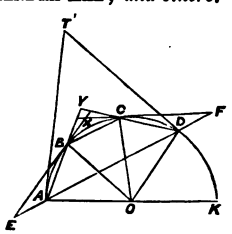
$$\Delta XEF = \sin \tfrac{1}{2}\theta \,(2 \sin \tfrac{1}{2}\theta + \sin \tfrac{3}{2}\theta)^2;$$

therefore $\Delta YAD : \Delta TAD : \Delta XEF$

$$= \sin\theta \sin^2 \tfrac{3}{2}\theta : \sin^3 \tfrac{3}{2}\theta : \sin \tfrac{1}{2}\theta \,(2 \sin \tfrac{1}{2}\theta + \sin \tfrac{3}{2}\theta)^2$$

$$= 1 : \frac{\sin \frac{3}{2}\theta}{\sin\theta} : \frac{\sin \frac{1}{2}\theta}{\sin\theta} \left(\frac{2 \sin \frac{1}{2}\theta}{\sin \frac{3}{2}\theta} + 1 \right)^2$$

= (in limit) $1 : \frac{3}{2} : \frac{1}{2}(\frac{2}{3} + 1)^2 = 18 : 27 : 35.$



6246. (By Prof. TOWNSEND, F.R.S.)—In the rotation, round its axis of figure, of a liquid spheroid of uniform density, held in permanent form by its own attraction for the ordinary law of the inverse square of the distance; show, from the equation of permanence, that

$$\frac{\omega^2}{4\pi\rho} = \frac{2}{3 \times 5}\lambda^2 - \frac{4}{5 \times 7}\lambda^4 + \frac{6}{7 \times 9}\lambda^6 - \frac{8}{9 \times 11}\lambda^8 + \&c.,$$

the several quantities involved having their usual significations.

Solution by the PROPOSER; W. J. C. SHARP, M.A.; *and others.*

The equation of permanence of form of the revolving fluid being, as is well known, $\dfrac{3\lambda + 2\epsilon\lambda^3}{3 + \lambda^2} - \tan^{-1}\lambda = 0$, where $\epsilon = \dfrac{\omega^2}{4\pi\rho}$,

the above follows from it immediately on the substitution for $\tan^{-1}\lambda$ of its development in ascending powers of λ. When λ is so small that all powers of it above the lowest may be neglected in the above series, then approximately $\lambda^2 = \frac{15}{8}\frac{\omega^2}{\pi\rho}$, its well-known approximate value in that case.

6309. (By Prof. TOWNSEND, F.R.S.)—A variable conic being supposed to touch in every position the four sides of a fixed quadrilateral circumscribed to a circle; show that its two axes of figure envelope the parabola which touches the three diagonals of the quadrilateral, and has for directrix the right line which bisects their three lengths.

I. *Solution by* C. TAYLOR, M.A.; Prof. GENESE, M.A.; *and others.*

Call the straight line which bisects the three diagonals L, M, N of the quadrilateral *the diameter of the quadrilateral.* This line is the centre-locus of all conics inscribed in the quadrilateral.

If $C\infty$ and $C\infty'$ be the axes of any one of the inscribed conics, the sides of the triangles $C\infty\infty'$ and LMN, which are self-polar to this conic, touch one *conic*, in this case necessarily a *parabola* as touching the line at infinity. The directrix of this parabola passes through C and through the centre of the circle (which is the orthocentre of the triangle LMN), and therefore coincides with the diameter of the quadrilateral. Hence the parabola is the same for all positions of C, and is that of the question.

II. *Solution by the* PROPOSER.

This property is an obvious consequence from the several following, all of which are well known, viz.:—(1) That the centres of figure of the entire system of conics inscribed to any quadrilateral in a plane lie all on the right line which bisects the three diagonals of the quadrilateral. (2) That the triangle determined by the three diagonals is self-reciprocal with respect to every conic of the system. (3) That, when (as in the question) a conic of the system is a circle, its centre of figure coincides in position with the orthocentre of the triangle, and lies consequently on the directrix of every parabola which touches its three sides. (4) That the two axes of figure of every conic to which a triangle in the same plane is self-reciprocal, are the two rectangular conjugate rays of the involution determined at its centre by the three connectors with the vertices, and by the three parallels to the sides of the triangle. And (5) That the two tangents, from any point in the plane of the triangle, to the inscribed parabola whose directrix passes through the point, are also the two rectangular conjugate rays of the same involution for the point. From these several properties combined, that of the question is an obvious and immediate consequence.

[From (3) and (4) above, it appears at once, as is often useful, that, a triangle being supposed self-reciprocal with respect to a conic in its plane, the two axes of figure of the conic are the two tangents to the parabola inscribed to the triangle whose directrix passes through the centre of the conic.]

6347. (By FREDERICK PURSER, M.A.)—A variable conic being supposed to touch four fixed right lines in a plane; show that its axes of figure envelop in general a curve of the third class, which breaks up, when the four lines touch a common circle, into the parabola of Question 6309 and the centre of the circle.

I. *Solution by* Dr. F. TIRELLI; F. D. THOMSON, M.A.; *and others.*

Il est facile de voir, que les axes des coniques inscrites dans un quadrilatère *abcd* déterminent sur la ligne droite à l'infini de leur plan une involution du deuxième ordre projectife à la ponctuée r des centres des mêmes coniques. Or, il est bien clair, que les joignantes les points d'une ponctuée aux extrêmes segments correspondants d'une involution de deuxième ordre projectife à elle-même enveloppent une courbe de troisième classe. Si l'on suppose cependant, que le quadrilatère *abcd* soit circonscrit à un cercle ayant le centre C, le faisceau des lignes droites perpendiculaires de centre C est un enveloppement de première classe, qui fait partie de l'enveloppement susdit de troisième classe; c'est que, en tel cas, l'enveloppement des axes des coniques inscrites dans le quadrilatère est une conique, qui, à cause des coniques exceptionnelles du système (coniques réduites à deux points) touche aux diagonales du quadrilatère, et dont le cercle directeur se réduit à la ligne droite r. C'est donc pourquoi cette conique est une parabole, qui a r pour directrice.

II. *Solution by the* PROPOSER.

More generally, the axes of a variable conic touching any four fixed lines in a plane envelop a line of the third class, which, when the four lines touch a circle, breaks up into the parabola of the question, and the centre of the circle. This may be easily shown as follows:—

If we denote, for convenience, by the terms "harmonic line" of a given line, the line which is the locus of its poles with respect to a system of conics touching four fixed lines, it is evident that we may define an axis of a conic of the system as a line perpendicular to its harmonic line. For its pole with regard to its own conic is the point at ∞ in the directions perpendicular to it. Let now the equation of a conic of the system referred to the three diagonals of the quadrilateral be $a\alpha^2+b\beta^2+c\gamma^2=0$. Let also $p\alpha+q\beta+r\gamma=0$ be the equation of the line which is the locus of centres of conics of the system, *i.e.*, the harmonic of the line ∞. Then the coordinates of the centre being $\frac{\sin A}{a}, \frac{\sin B}{b}, \frac{\sin C}{c}$, a, b, c will be connected by the relations $\frac{p\sin A}{a}+\frac{q\sin B}{b}+\frac{r\sin C}{c}=0$.

Now, in general the pole of any line $l\alpha+m\beta+n\gamma=0$ is $\frac{l}{a}, \frac{m}{b}, \frac{n}{c}$.

The locus of such poles as the harmonic line is then

$$\frac{p\sin A}{l}\alpha+\frac{q\sin B}{m}\beta+\frac{r\sin C}{n}\gamma=0.$$

The condition that this line should be perpendicular to $l\alpha + m\beta + n\gamma = 0$, is $p \sin A + q \sin B + r \sin C - \Sigma \left(\frac{qn \sin B}{m} + \frac{rm \sin C}{n} \right) \cos A = 0$,

or $\Sigma l \cos A (rm^2 \sin C + qn^2 \sin B) - lmn (p \sin A + q \sin B + r \sin C) = 0$.

The envelop of a line perpendicular to its harmonic is then, in general, a curve of the third class.

Now, we have, identically,

$$\Sigma l \cos A (rm^2 \sin C + qn^2 \sin B) - lmn (p \sin A + q \sin B + r \sin C)$$
$$= (l \cos B \cos C + m \cos C \cos A + n \cos A \cos B) (mnp \tan A + nlq \tan B + lmr \tan C)$$
$$- \frac{lmn}{\sin A \sin B \sin C} \left(\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} \right).$$

If, now, the quadrilateral be circumscribable to a circle, the centre of this circle will be the orthocentre of the triangle of reference.

Expressing, then, that the centre of the circle lies on the general locus of centres, we have $\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} = 0$.

The general curve of the third class above breaks up then in this case into two factors, one being the point equation of the orthocentre, the other the tangential equation of a conic touching the sides of the triangle of reference, and which in virtue of the equation

$$\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} = 0,$$

is touched also by $\alpha \sin A + \beta \sin B + \gamma \sin C = 0$, *i. e.*, is a parabola.

It is easy to see the geometrical reason for the resolution of the general curve into factors which takes place in this case. In fact, when one of the conics of the system is a circle, any line through its centre is an axis of the circle, and therefore a line of the envelop, which must therefore consist of the centre and a conic.

[The general result given above is in accordance with that of Quest. 6290, (whereof a solution is given on p. 34 of this volume,) which is a particular case of it. Here one of the four fixed lines is the line ∞. The curve of the third class is the tricusp which is the envelop of the finite axis; the line at ∞, which here constitutes the other axis of conics of the system, touching the envelop, as it should do, being, in fact, the bitangent to it at the point I, J.]

6169. (By Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR.

Solution by Professor Seitz, M.A.

Let $OP = x$, $OQ = y$, $\angle POQ = \theta$, and let z = the perpendicular distance of R from OPQ, and r = the radius of the sphere. Then $\frac{1}{6} ryz \sin \theta$ = the volume of the tetrahedron; an element of the sphere at P is $4\pi x^2 dx$, at Q

it is $2\pi y^2 \sin\theta\, dy\, d\theta$, and at R it is $\pi(r^2-z^2)\, dz$. The limits of each of the variables x, y, z are 0 and r, and those of θ are 0 and π.

Hence the average volume of the tetrahedron is

$$\frac{\int_0^\pi\int_0^r\int_0^r\int_0^r \tfrac{1}{6}xyz\sin\theta\,.\,8\pi^3\sin\theta\, d\theta\, x^2\, dx\, y^2\, dy\,(r^2-z^2)\, dz}{\int_0^\pi\int_0^r\int_0^r\int_0^r 8\pi^3\sin\theta\, d\theta\, x^2\, dx\, y^2\, dy\,(r^2-z^2)\, dz}$$

$$= \frac{9}{8r}\int_0^\pi\int_0^r\int_0^r\int_0^r \sin^2\theta\, d\theta\, x^3\, dx\, y^3\, dy\,(r^2-z^2)\, z\, dz$$

$$= \frac{9}{32r^5}\int_0^\pi\int_0^r\int_0^r \sin^2\theta\, d\theta\, x^3\, dx\, y^3\, dy$$

$$= \frac{9}{128r}\int_0^\pi\int_0^r \sin^2\theta\, d\theta\, x^3\, dx = \frac{9r^3}{512}\int_0^\pi \sin^2\theta\, d\theta = \frac{9\pi r^3}{1024}.$$

[Mr. Monro, in his Solution of this Quest. (*Reprint*, XXXIII., 75), "errs," says Prof. Seitz, "in limiting P to a semi-diameter, and Q to a semi-circle." In reference to this correction, Mr. Monro writes as follows:—"I certainly erred, as Mr. Seitz says. The ranges of P and Q should not have been a line and surface at all, but a volume in each case, namely, the infinitesimal cone and wedge of which the semi-diameter and semi-circle will give the finite dimensions. The distances of the mean positions will thus be $\frac{3}{4}$, $\frac{3}{16}\pi$, $\frac{3}{8}$ of the radius; and $\frac{1}{8}$ of the product is $\frac{9}{1024}\pi$.]

5901. (By Prof. Sylvester, F.R.S.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

6094. (By Prof. Sylvester, F.R.S.)—1. If the process of residuation, described in Question 2391, be applied to a quartic instead of to a cubic curve, prove that the minimum number of points in any *residuum* is 3, that being the number applicable to the case of an odd number of initial points. If the initial number is of the form $4i$, show that the minimum number in a residuum is 4, and when the initial number is of the form $4i+2$, 6, except for the case when $i=0$, when it is obviously only 2.

2. Prove that, if the number of initial points is odd, the 3 *residuals* are functions only of those points.

3. In general, if p be prime to μ and p initial points be taken in a curve of the μ^{th} order, prove that the residuum may be reduced to consist of $\frac{1}{2}(\mu-1)(\mu-2)$ residual points, which residuals are all functions of the p given points only, being independent of the particular process of residuation applied to arrive at them.

Solution by W. J. CURRAN SHARP, M.A.

The principle of residuation, which is enunciated in these two theorems, depends upon the proposition that "every curve of the n^{th} degree, which is drawn through $np - \frac{1}{2}(p-1)(p-2)$ points on a curve of the p^{th} degree (p less than n), meets this curve in $\frac{1}{2}(p-1)(p-2)$ other fixed points." [See SALMON's *Higher Plane Curves*, p. 18.]

If $p = 3$, there is only one additional point.

First, if $\mu = 3i-1$, U_i an i-ic will cut in one additional point A.

The ν-ic U_ν will cut in the μ points, and μ_1 other points,
ν_1-ic U_{ν_1} ,, μ_1 ,, μ_2 ,,
&c. &c. &c. &c.
ν_{2m}-ic $U_{\nu_{2m}}$,, μ_{2m}, and one point B,

and $i+\nu_1+\ldots+\nu_{2m-1} = \nu+\nu_2+\ldots+\nu_{2m} \equiv r$ suppose,

and the r-ic $U_i U_{\nu_1} \ldots U_{\nu_{2m-1}}$ will intersect the cubic in the points μ, μ_1, &c., and A; and the r-ic $U_\nu . U_{\nu_2} \ldots U_{\nu_{2m}}$ will intersect it in the same $3r-1$ points (the μ, μ_1, &c. points) and B; hence A and B are identical; and similarly if $\mu = 3i+1$.

If $p = 4$, the number of additional points is 3, a value which can only be obtained if μ be odd; since $\mu_1 = 4\nu-\mu$, $\mu_2 = 4\nu_1 - 4\nu + \mu$, &c. Hence also, if $\mu = 4i$, all the residua are of the same form, and if $\mu = i+2$, $\mu_1 > 2$ unless $\nu < i+2$.

If the number of points μ be odd, so that a residuum of 3 is possible, it can be proved exactly as in the case of the cubic that any two residua consisting of three points are identically the same. And so also in the general case.

[Like the corresponding proposition for cubics, the extension in Quest. 6094 includes a vast number of geometrical properties. In No. 1 of Vol. 3 (1880) of the *American Journal of Mathematics* will be found an application of the Theory of Residuation to the determination of all the points on a Cubic Curve rationally connected with any given point. These points, which usually are *infinite*, become for particular positions of the initial point periodic, and consequently *finite* in number. Such positions of the initial point may be termed *sub-inflexions*. Any such sub-inflexion is a point one of whose rational derivatives is a point of inflexion on the conic. The simplest instance of sub-inflexions is afforded by the 27 points of Plücker, viz., the anti-tangentials of the 9 points of inflexion.]

6383. (By J. VENN, M.A.)—Are there any inconsistencies or redundancies in the following rules:—(1) The *Financial* Committee shall be chosen from among the *General* Committee; (2) No one shall be a member

both of the *General* and *Library* Committees unless he be also on the *Financial* Committee; (3) None of the *Library* Committee shall be on the *Financial*?

I. *Solution by the* Rev. R. HARLEY, F.R.S.; Prof. MATZ, M.A.; *and others.*

The given rules contain no inconsistencies, but they may be simplified and reduced by striking out the third rule and the last clause of the second. This becomes obvious on writing them in the Boolian forms.

Let F, G, L stand for member of the *Financial*, *General*, and *Library* Committees respectively, then we have

$$F = FG, \quad GL = FGL, \quad FL = 0 \ldots\ldots\ldots\ldots\ldots (1, 2, 3).$$

Now, (1) and (2) give $GL = FL$, and therefore, by (3), $GL = 0$.....(4). But this is given by (2) and (3); thus, $FGL^2 = FGL \cdot 0$; or $FGL = 0$, and therefore, by (2), $GL = 0$.

Replace then the second and third rules by the following:—

(4) None of the *Library* Committee shall be on the *General*.

Rules (1) and (4) are exactly equivalent in force to the three given rules (1), (2), (3). For we have seen that (2) and (3) give (4), and it may readily be shown that (1) and (4) give (1), (2), and (3); thus,

$$FGL = FG \cdot 0 = 0 = GL,$$

which is (2), and, by (4), $L = \frac{0}{G} = \frac{0}{0}(1-G)$;

therefore, by (1), $FL = \frac{0}{0}FG(1-G) = 0$, which is identical with (3).

II. *Solution by* H. MCCOLL, B.A.; Prof. GENESE, M.A.; *and others.*

Speaking of any originally unclassed person, let f, g, l respectively denote the statements: He shall be on the *Financial* Committee; He shall be on the *General* Committee; He shall be on the *Library* Committee. The three rules will then be expressed by the compound implication $(f : g)(gl : f)(l : f')$, which is equivalent to the implication $fg' + glf' + lf : 0$. Reducing the antecedent of this zero implication to its primitive form, we get $fg' + gl : 0$, which is equivalent to $(f : g)(g : l)$, and may be read: Everyone on the Financial Committee shall be on the General Committee, and no one on the General Committee shall be on the Library Committee.

6075. (By DONALD MCALISTER, D.Sc. — A sphere full of water, is hung up by a string. Suddenly a slender uniform crack extends from one end of the vertical diameter to the other. Prove that the time of emptying (on the hypothesis of "parallel sections") is to the time of free fall from rest down the diameter, as the area of the sphere is to that of the lune.

Solution by J. J. WALKER, M.A.

The same diameter remaining vertical during discharge, and the jet being uncontracted, let θ be the angle of the slit, z the height of the free

surface above the centre at the time t. In the element of time dt, the quantity discharged will be equal to

$$(2g)^{\frac{1}{2}} r\theta\, dt \int_{-r}^{r} (z-x)^{\frac{1}{2}}\, dx, \text{ or } \tfrac{2}{3}(2g)^{\frac{1}{2}} r\theta\, (z+r)^{\frac{3}{2}}\, dt,$$

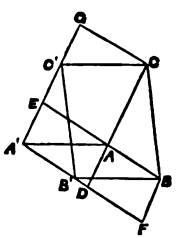
x and $x+dx$ being the distances from the centre of two horizontal planes intercepting an element of the slit. Equating this with the diminution of volume $\pi(r^2-z^2)\,dz$, there results for the time of emptying

$$t = \tfrac{2}{3}\pi \int_{-r}^{r} (r-z)\,(r+z)^{-\frac{1}{2}}\, dz \div (2g)^{\frac{1}{2}} r\theta = 4\pi (2r)^{\frac{3}{2}} \div (2g)^{\frac{1}{2}}\theta = 2\pi t' : \theta,$$

if t' is the time of falling freely down the height $2r$.

6196. (By the Editor.)—If ABC, A′B′C′ be two triangles with their corresponding sides equal and parallel; prove that (1) of the parallelograms formed by joining corresponding vertices one is equal to the other two; and hence (2) prove Euc. I. 47.

Solution by R. Rawson; C. Morgan, B.A.; *and others.*



The figures A′B, A′C, B′C are parallelograms; also A′C′CBB′A′ = B′C + A′B′C′ = A′B + A′C + ABC, and $\triangle ABC = A'B'C'$; therefore

$$B'C = A'B + A'C \quad \ldots\ldots\ldots\ldots (1).$$

It is readily seen that

$$B'C = BD + CE \text{ (Euc. I. 35)} \quad \ldots\ldots\ldots\ldots (2),$$

which seems to be necessary to prove Euc. I. 47 by the application of the neat property enunciated in (1). It is quite true that the form and position of the triangles ABC, A′B′C′ can be so constructed as to insure the parallelograms B′C, BD, CE to be the squares upon BC, AB, AC respectively. If this construction could be seen at a glance by an ordinary student of Euclid, then the proof of Euc. I. 47, by the above method, would be complete, and exceedingly simple.

Make the angle CAB a right angle, and the $\triangle ABC$ equal to the $\triangle BB'F$ in every respect, and draw AA′ equal and parallel to BB′ = BC.

Now, it is easily seen that $\triangle A'B'C' = \triangle ABC$; and that the parallelograms BD, CE are the squares upon AB, AC as required.

Because $\angle B'BF = CBA$, and $\angle BB'F = ABB' = ACB$;

therefore $\angle ABB' + CBA = CBA + ACB =$ a right angle $= \angle B'BC$;

therefore B′C is a square on BC; and (2) becomes $BC^2 = AB^2 + BC^2$.

The following is a simple application of the property in the question: Produce AC to D, making AD = AB; and AB to E, making AE = AC; through B draw FH equal and parallel to CD; upon HF draw the square A′H; and make FB′ = A′C′ = GC = AC.

Then we have A′B = GC′ = CH = AB, and the figure B′C′CB, being

equilateral and also rectangular, is the square on BC; hence,

square A'H = square B'C + 4ΔBAC = square AB + square AC + 4ΔBAC;

therefore $BC^2 = AB^2 + AC^2$.

Hence, square A'H = square AD + square AC + 2AH,

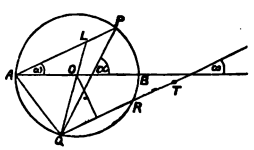
or $CD^2 = AD^2 + AC^2 + 2AC \cdot AD$. (Euc. II. 4.)

5600. (By Christine Ladd.)—Required the envelop of the Simson (or pedal) line.

6290. (By the Rev. F. D. Thomson, M.A.)—Find the envelop of the axis of a parabola inscribed in a given triangle.

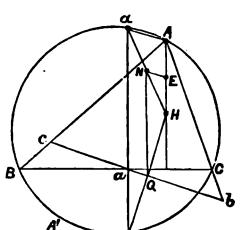
Solution by R. F. Davis, M.A.; G. Eastwood, M.A.; *and others.*

1. The consideration of many theorems in which the three-cusped hypocycloid appears either as a locus or an envelop may be simplified by the following theorem:—"If A be a fixed point on a given circle, and PQ a chord having a fixed direction, the envelop of QR drawn parallel to AP is a three-cusped hypocycloid concentric with and circumscribing the circle." Let AOB be the diameter through A; ω, α the inclinations to it of AP (or QR) and PQ. Let OQ meet AP in L; then $\angle OQR = \angle ALQ = \angle AOQ - \omega = 2\alpha - 3\omega$. Thus the tangential polar equation to the envelop of QR is $p = a \sin(2\alpha - 3\omega)$, which represents a three-cusped hypocycloid, concentric with and circumscribing the given circle.



A remarkable addition to this theorem is the following:—"If along QR points T, T' be taken such that QT = QT' = 2a, then T, T' are situate on the same same curve." This may be shown either by analysis or geometry, and leads at once to the known property that the part of the tangent to a three-cusped hypocycloid intercepted by the curve is of constant length, and its middle point lies on the inscribed circle.

2. Let S be a point on the circle circumscribing the triangle ABC; Sa, Sb, Sc perpendiculars upon the sides. Then abc is termed the Simson (or pedal) line corresponding to S (*Reprint*, Vol. XXIX., p. 80); and if a be the point where Sa again meets the circle, and H the orthocentre, it is parallel to Aa, and bisects SH in Q. Bisect HA, Ha in E, N; then E, N, Q are points on the nine-points circle. Therefore, by the preceding theorem, since E is a fixed point, and QN (parallel to Sa) a fixed direction, the envelop of abc (parallel to EN) is a three-cusped hypocycloid concentric with and circumscribing the nine-points circle.

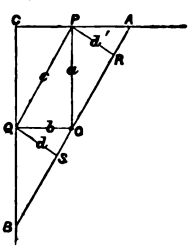


3. The Simson line being, *ipso facto*, the tangent at the vertex of the parabola inscribed in the triangle having S for its focus, then, if A′ be the extremity of the diameter through A, the axis of the parabola will be the straight line through S parallel to A′a. Therefore, A′ being a fixed point, and Sa a fixed direction, the envelop of the axis will be a three-cusped hypocycloid concentric with and circumscribing the circle about the triangle. [See p. 29 of this Volume of the *Reprint*.]

6226. (By Professor MATZ, M.A.) — A post is a feet from the north wall, and b feet from the west wall, of a room whose sides meet each other at right angles; find the sides of the largest rectangular table that can be passed so as to touch at the same time the walls and the post, the surface of the table to remain horizontal all the time.

Solution by Prof. GENESE, M.A.; R. TUCKER, M.A.; *and others.*

Let the ground be the plane of (x, y), the axes of x and y being on the west and north walls respectively; and let (u, v) be the sides of a rectangle PQSR with the ends of u (P, Q) on x and y; then RS will envelop a curve obtained by eliminating α from



$$x\cos\alpha + y\sin\alpha = u\sin\alpha\cos\alpha + v,$$

$$-x\sin\alpha + y\cos\alpha = u\cos 2\alpha.$$

The condition that RS should pass round between the walls and the point O (b, a), is that O should *not* lie between the curve and the axes; or to get the greatest rectangle that O should lie on this curve;

$$\therefore\ b\cos\alpha + a\sin\alpha = u\sin\alpha\cos\alpha + v,\quad -b\sin\alpha + a\cos\alpha = u\cos 2\alpha \ldots (1, 2).$$

The elimination of α gives the relation between u and v.

Again, uv is to be a maximum, therefore $\frac{du}{u} + \frac{dv}{v} = 0$. From (1),

we have $\quad du\sin\alpha\cos\alpha + dv + u\cos 2\alpha = -b\sin\alpha + a\cos\alpha;$

or, using (2), $du\sin\alpha\cos\alpha + dv = 0$, therefore $u\sin\alpha\cos\alpha - v = 0 \ldots\ldots (3)$.

From (1), (2), (3), we easily find $u = (a^2+b^2)^{\frac{1}{2}}$, $v = ab(a^2+b^2)^{-\frac{1}{2}}$.

Thus the length of the table is equal to the distance from the post to the intersection of the walls; and the area of the table is equal to the area enclosed, on the ground, by the walls and planes and parallels to them through the post.

[Let CA, CB be the walls of the room, and O the position of the post; then, drawing OP, OQ perpendicular to the walls, AB parallel to PQ, and PR, QS perpendicular to AB, the rectangle PQSR will be the required slab of table in its critical position. For, PQSR being the maximum rectangle, having its vertices P, Q on the sides CA, CB, that could be inscribed in the triangle ABC, no slab of greater area could attain to, much less pass through, the position in the figure. The position being supposed attained by the slab PQSR, then, on account of the perpendicularity of OP, OQ to

CA, CB, a small rotation in either direction of the slab round O as centre would disengage P and Q simultaneously from contact with the walls, and disengage, in consequence, the slab from its critical position.

Denoting by (c, d) the length and breadth of the slab, and by (a, b) the coordinates of the post, we have, evidently,

$$c = (a^2+b^2)^{\frac{1}{2}}, \quad d = ab \,.\, c^{-1} = ab\,(a^2+b^2)^{-\frac{1}{2}},$$

which give at once c and d in terms of a and b.]

6391. (By J. J. Walker, M.A.)—If O, A, B, C, D are any five points in space, prove that lines drawn from the middle points of BC, CA, AB respectively parallel to the connectors of D with the middle points of OA, OB, OC, meet in one point E, such that DE passes through, and is bisected by, the centroid of the tetrahedron OABC. [Quest. 6220 is a special case, in *two* dimensions, of the foregoing theorem in *three* dimensions.]

I. *Solution by* Professor Minchin, M.A.

Taking O as origin of vectors, and denoting the vectors to A, B, C, D by $\alpha, \beta, \gamma, \delta$ respectively, the equations of the three lines which are to be proved concurrent are

$$\rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(x-1)\,\alpha - x\delta, \quad \rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(y-1)\,\beta - y\delta,$$
$$\rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(z-1)\,\gamma - z\delta.$$

But, if we make $x = y = z$, these are all satisfied by the vector $\frac{1}{2}(\alpha+\beta+\gamma) - \delta$. If this vector is ϖ, we have $\frac{1}{2}(\varpi+\delta) = \frac{1}{2}(\alpha+\beta+\gamma) =$ vector to centroid of tetrahedron OABC.

II. *Solution by* Professor Genese, M.A.

Let P, Q, R be the middles of BC, CA, AB; L, M, N the middles of the connectors; then the sides of the triangle LMN are equal and parallel to those of PQR. Therefore parallels through PQR to DL, DM, DN will meet in a point E such that EP = DL, EQ = DM, ER = DN; also EP must be parallel to DL; therefore DLEP is a parallelogram, and DE is bisected at the middle point of PL.

6173. (By Hugh McColl, B.A.)—If the statements a, b, f are true, then the statements c and x are true, or else the statements d, e, y are true. If the statement a is true, f false, and y true, then c is true and x false, or else d is false and e true. What inference may be drawn from these premises with respect to the truth or falsehood of the statements a, b, c, d, e, f, eliminating the statements x and y?

Solution by C. J. Monro, M.A.

Put A for $a\,(1-d'e)$; then, in Boole's notation (with accents),

$$fab\,(1-cx)\,(1-dey) = 0, \quad Af'\,(1-cx')\,y = 0.$$

The left-hand sides are positive class-terms, and may be combined by addition. Add them, making x and y successively 1 and 1, 1 and 0, 0 and 1, 0 and 0. The results are $fabc'(1-de)+Af'$, $fabc'$, $fab(1-de)+Af'c'$, fab. Take their product: this gives $fabc'(1-de)=0$ for the result of elimination. That is the whole effect of the data, irrespectively of x and y; but of course it may be expressed in various forms of inference. Perhaps the most natural is that, if f, a, and b are true, then c is true, or d and e, or all three. In fact, if we develope for fab in terms of c and de, we get $fab = \S c + \S c'de$.

6037. (By Prof. CROFTON, F.R.S.)—Prove that—

$$e^{x+D}\,F(x) = e^1 e^x\,F(x+1), \quad \text{and} \quad e^{a(x+D)}\,Fx = e^{1a^2}e^{ax}\,F(x+a). \ \ldots$$

Solution by W. J. C. SHARP, M.A.; G. HEPPEL, M.A.; *and others.*

If (as in BOOLE's *Diff. Eq.*, 1st Ed., p. 395) $x+D$ be denoted by π, and D be resolved into D_1 and D_2, operating on x only as entering into $F(x)$ and π, and if $x+D_1 \equiv \pi_1$; then we have

$$\pi^n = \pi_1{}^n + \frac{n(n-1)}{1}\cdot\frac{\pi_1{}^{n-2}}{2} + \frac{n(n-1)(n-2)(n-3)}{1.2}\cdot\frac{\pi_1{}^{n-4}}{4} + \ldots + \frac{n!}{r!(n-2r)!}\cdot\frac{\pi_1{}^{n-2r}}{2^r};$$

therefore

$$\begin{aligned} e^{a\pi} &= 1 + \frac{a\pi}{1} + \frac{a^2\pi^2}{2!} + \frac{a^3\pi^3}{3!} + \ldots + \frac{a^n\pi^n}{n!} + \&c. \\ &= 1 + \frac{a\pi_1}{1} + \frac{a^2\pi_1{}^2}{2!} + \frac{a^3\pi_1{}^3}{3!} + \ldots + \frac{a^n\pi_1{}^n}{n!} + \&c. \\ &\quad + \frac{a^2}{2} + \frac{a^3\pi_1}{1}\cdot\frac{1}{2} + \ldots + \frac{a^n\pi_1{}^{n-2}}{n-2!}\cdot\frac{1}{2} + \&c. \\ &\quad + \&c.\ \&c. \qquad + \frac{a^n\pi_1{}^{n-4}}{(n-4)!\,2!}\cdot\frac{1}{2^2} + \&c. \\ &= \left(1 + \frac{a^2}{2} + \frac{1}{2!}\cdot\frac{a^4}{2^2} + \&c.\right)\left(1 + \frac{a\pi_1}{1} + \frac{a^2\pi_1{}^2}{2!} + \&c.\right) = e^{1a^2}\cdot e^{a\pi_1}; \end{aligned}$$

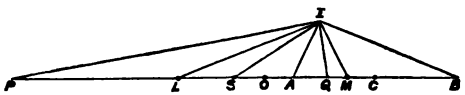
therefore $e^{a\pi}\,F(x) = e^{1a^2}\cdot e^{a\pi_1}\cdot F(x) = e^{1a^2}e^{x}e^{aD_1}\,F(x) = e^{1a^2}e^{ax}\,F(x+a)$; and, if $a=1$, this becomes $e^{\pi}\,F(x) = e^1 e^x\,F(x+1)$.

[For Prof. CROFTON's solution, see a paper in the *Quarterly Journal of Mathematics* for October, 1879.]

6386. (By W. S. MCCAY, M.A.)—If two circles cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar to one of the circles as the other point has to the other circle; and (2) the distance between such a pair of points subtends a right angle at an intersection of the circles.

Solution by Prof. GENESE, M.A.; H. MURPHY; *and others.*

Let the circles AIB, LIM, with centres C and O, cut at right angles; and let S be any point in COL, and P and Q its conjugate points with respect to LIM and AIB. Then $\angle SIP = 2SIL$, $\angle SIQ = 2SIA$. Therefore $\angle PIQ = 2LIA = LIM =$ a right angle. Conversely, if PIQ be a right angle, the polar of P with respect to one circle, and the polar of Q with respect to the other, will coincide.



6331. (By J. W. RUSSELL, M.A.)—From the centres α, β, γ of the circles escribed to the triangle ABC, are drawn perpendiculars to the sides produced, so as to form a hexagon $\alpha\gamma'\beta\alpha'\gamma\beta$ whose opposite sides are parallel; show that the perpendiculars from α', β', γ' on the corresponding sides of the triangle meet in a point.

Solution by F. D. THOMSON, M.A.; J. O'REGAN; *and others.*

Here αA, βB, γC are the three perpendiculars of the triangle $\alpha\beta\gamma$. Let them meet in O. Draw $\alpha\beta'$ perpendicular to AB, $\gamma\beta'$ perpendicular to BC, &c.

Then, by the geometry,

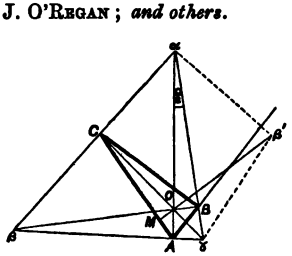
$$\angle\gamma\alpha\beta' = \alpha\gamma\beta' = \tfrac{1}{2}B,$$

also $$\alpha O\gamma = \tfrac{1}{2}\pi + \tfrac{1}{2}B,$$

and hence β' is the centre of the circle round $\alpha O\gamma$. Hence

$$\angle\beta' O\gamma = \beta'\gamma O = \tfrac{1}{2}(\pi - C).$$

But if $\beta' O$ meet CA in M, $\angle\beta' O\gamma = COM$.



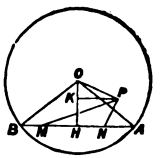
Therefore $\beta' OM$ is perpendicular to CA. Similarly, $\alpha' O$ is perpendicular to BC, and $\gamma' O$ is perpendicular to AB. Therefore the three perpendiculars from $\alpha'\beta'\gamma'$ on the corresponding sides of ABC meet in a point.

6319. (By W. S. B. WOOLHOUSE, F.R.A.S.) — On a given circle, of radius unity, a corner of a triangle is taken at a given distance (c) from the centre; the other two corners are taken at random on the surface; prove that the average area of all such triangles, in parts of the area of the circle, is $\frac{1}{\pi^2}(\frac{4}{3} + \frac{2}{3}c^2 - \frac{1}{3}c^4 + \frac{1}{36}c^6)$.

Solution by Professor SEITZ, M.A.

Let MNP be the triangle, P being fixed, AB the chord through M, N,

and O the centre of the circle. Draw OH perpendicular to AB, and PK to OH. Let



$OP = c$, $OA = 1$, $AM = x$, $MN = y$, $\angle AOH = \theta$,

$\angle POH = \phi$, $\theta' = \cos^{-1}c$, and $\cos\theta = c\cos\psi$.

Then we have area $MNP = \frac{1}{2}(\cos\theta - c\cos\phi)\,y$, when O and P lie on the same side of AB, and $= \frac{1}{2}(c\cos\phi - \cos\theta)\,y$, when they lie on opposite sides. An element of surface at M is $\sin\theta\,d\theta\,dx$, and at N it is $y\,d\phi\,dy$. When $\theta > \theta'$, O and P will lie on the same side of AB for all values of ϕ from 0 to π; but when $\theta < \theta'$, O and P will lie on opposite sides of AB from $\phi = 0$ to $\phi = \psi$, and on the same side from $\phi = \psi$ to $\phi = \pi$. The limits of x are 0 and $2\sin\theta$; and of y, 0 and x, and doubled. The results of the integrations with respect to ϕ must be doubled.

Hence, since the whole number of triangles is π^2, and the area of the circle is π, the average area is

$$\frac{4}{\pi^3}\int_0^{\theta'}\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{x}\tfrac{1}{2}(\cos\theta - c\cos\phi)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy$$

$$+\frac{4}{\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{2\sin\theta}\int_0^{x}\tfrac{1}{2}(c\cos\phi - \cos\theta)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy$$

$$+\frac{4}{\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\int_{\psi}^{\pi}\int_0^{2\sin\theta}\int_0^{x}\tfrac{1}{2}(\cos\theta - c\cos\phi)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy$$

$$=\frac{8}{3\pi^3}\int_0^{\theta'}\int_0^{\pi}(\cos\theta - c\cos\phi)\sin^5\theta\,d\theta\,d\phi$$

$$+\frac{8}{3\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\left\{\int_0^{\psi}(c\cos\phi - \cos\theta)\,d\phi + \int_{\psi}^{\pi}(\cos\theta - c\cos\phi)\,d\phi\right\}\sin^5\theta\,d\theta$$

$$=\frac{8}{3\pi^3}\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos\theta\,d\theta + \frac{16c^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}(\sin\psi - \psi\cos\psi)(1-c^2\cos^2\psi)^2\sin\psi\,d\psi$$

$$=\frac{1}{\pi^2}\left(\tfrac{4}{5} + \tfrac{8}{9}c^2 - \tfrac{1}{8}c^4 + \tfrac{1}{288}c^6\right).$$

If all the corners of the triangle are taken at random on the surface of the circle, the average area of all the triangles is

$$\int_0^1 \frac{1}{\pi^2}\left(\tfrac{4}{5} + \tfrac{8}{9}c^2 - \tfrac{1}{8}c^4 + \tfrac{1}{288}c^6\right)c\,dc + \int_0^1 c\,dc = \frac{35}{48\pi^2}.$$

6298. (By A. Martin, M.A.)—If three equal circles be piled up at random on a horizontal plane, prove that the probability that the pile will stand, is

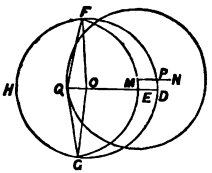
$$= \frac{1}{16} + \frac{3\sin^{-1}\frac{1}{4}}{8\pi} - \frac{9\sqrt{15}}{256\pi}.$$

Solution by Professor Seitz, M.A.

Let DFHG be the first circle, O its centre, M and N the centres of the

second and third circles. The pile will stand, if M and the middle point P of MN are in the circle O, and N is in the circle M.

If the circle M be moved parallel to itself so that MN will be constantly bisected by the circumference of the circle O, M will describe the arc GEF, whose centre is Q, and the area of GEFH will represent the number of favourable positions.



Let OD = 1, MN = x, $\angle$FOQ = θ, and let ϕ = the angle which MN makes with some fixed line. Then OQ = $\frac{1}{2}x = 2\cos\theta$, $x = 4\cos\theta$, $dx = -4\sin\theta\, d\theta$, and area GEFH = $2(\theta - \sin\theta\cos\theta)$.

The limits of x are 0 and 1; of θ, $\cos^{-1}\frac{1}{4} = \theta'$ and $\frac{1}{2}\pi$; and of ϕ, 0 and 2π. Hence, since the whole number of positions of the three circles is $16\pi^2$, the required probability is

$$\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi} 2(\theta - \sin\theta\cos\theta)\, x\, dx\, d\phi$$

$$= \frac{4}{\pi}\int_{\theta'}^{\frac{1}{2}\pi} (\theta - \sin\theta\cos\theta)\sin\theta\cos\theta\, d\theta = \text{the result stated.}$$

6213. (By A. Martin, M.A.)—A cylinder, of radius r, *rolls* down the surface of another one, radius R, resting on a horizontal plane, the surface of both cylinders and plane being rough enough to secure perfect rolling. Determine the circumstances of the motion, the point of separation, and the path of the axis of the upper cylinder.

Solution by J. J. Walker, M.A.

Suppose C, C′, points in which the axes are cut by a plane perpendicular to their common direction, to be the centres of gravity of the cylinders, and the normal and tangential actions, N, T, of the lower on the upper, as also those, N′, T′, of the horizontal plane, on which the lower rolls, on it, to lie in that same vertical plane. Let θ be the angle which the line CC′ makes with the horizontal; ϕ, ϕ' the angle through which the cylinders have revolved, in the time t. Take the origin in the line of intersection of the vertical and horizontal planes above defined; and let xy be the co-ordinates of C, x'R those of C′. Then we have

$$x - x' = (r+\mathrm{R})\cos\theta,\ y = \mathrm{R} + (r+\mathrm{R})\sin\theta,\ \dot{x}' = -\mathrm{R}\dot{\phi}',\ r\dot{\phi} = \mathrm{R}\dot{\phi}' \quad \ldots\ldots(1).$$

The movement of the upper cylinder is defined by

$$m\ddot{x} = \mathrm{N}\cos\theta - \mathrm{T}\sin\theta,\ m\ddot{y} = -mg + \mathrm{N}\sin\theta + \mathrm{T}\cos\theta,\ mk^2\ddot{\phi} = r\mathrm{T} \quad \ldots\ldots(2),$$

that of the lower by

$$m'\ddot{x}' = -\mathrm{N}\cos\theta + \mathrm{T}\sin\theta + \mathrm{T}',\ m'k'^2\ddot{\phi}' = \mathrm{R}(-\mathrm{T} + \mathrm{T}') \quad \ldots\ldots(3).$$

From these three systems of equations, we get ($u = \cos\theta$)

$$\left\{(m+m')r^2\mathrm{R}^2 + \mathrm{R}^2mk^2 + r^2m'k'^2\right\}\ddot{x} = (r+\mathrm{R})\left\{\mathrm{R}^2mk^2 + r^2m'(k'^2+r^2)\right\}\ddot{u},$$

whence

$$\{\ \}x = \{\ \}\cos\theta + ct + c';\ \text{and}\ (r+\mathrm{R})\left\{mr^2[k^2 - r^2(1-u^2)^{\frac{1}{2}}]\ddot{u} - \mathrm{A}\ddot{\theta}\right\} = \mathrm{A}g,$$

where A stands for the coefficient of $\ddot{x}$ above. This latter equation, involving both $\ddot{\theta}$ and $(\dot{\theta})^2$, is insoluble actually; but, supposing it to give

$\theta = F(t)$, x and y would be determined as functions of the parameter t, and the locus of C be therefore found implicitly. Finally, substituting for $\ddot{x}, \ddot{y}, \theta$, their values, in terms of t, in $\ddot{x}\cos\theta + (\ddot{y}+g)\sin\theta = 0$, the instant of separation of the cylinders would be determined.

6404. (By Sir James Cockle, F.R.S.)—Transform

$$y'' + 2(A\cot x + E\tan x)\,y' + \{Æ^2 - (A-E)^2\}\,y = 0 \ldots\ldots\ldots\ldots(1)$$

into an equation of the form called by Boole binomial.

I. *Solution by* J. Hammond, M.A.; W. B. Grove, M.A.; *and others.*

The equation $y'' + 2(A\cot x + E\tan x)\,y' + Cy = 0$

is transformed into a binomial equation by taking the independent variable to be either $\cos x$ or $\sec x$, or any power of $\cos x$ or $\sec x$.

1. Writing $\cos x = X$, we have

$$y' = -\sin x\frac{dy}{dX},\quad y'' = -X\frac{dy}{dX} + (1-X^2)\frac{d^2y}{dX^2},$$

and the given equation becomes

$$(1-X^2)\frac{d^2y}{dX^2} - X\frac{dy}{dX} - 2\left(AX + \frac{E(1-X^2)}{X}\right)\frac{dy}{dX} + Cy = 0;$$

which, by assuming $X = e^\theta$, is reduced to the form

$$y - \frac{(D+A-E-2)^2 - Æ^2}{D(D-1-2E)}\,e^{2\theta}y = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

2. Writing $\sec x = X$, we have

$$y' = X\frac{dy}{dX}\tan x,\quad y'' = X^2\frac{dy}{dX} + (X^2-1)\left(X\frac{d}{dX}\right)^2 y,$$

or, if $X = e^\theta$, $y' = \tan x\,Dy$, $y'' = (e^{2\theta}-1)\,D^2y + e^{2\theta}\,Dy$; and the equation becomes $(e^{2\theta}-1)\,D^2y + e^{2\theta}\,Dy + 2\{(A+E)(e^{2\theta}-1)\}\,Dy$

$$+ \{Æ^2 - (A-E)^2\}\,y = 0,\quad \text{or}\quad y - \frac{(D-2)(D-1+2E)}{(D-A+E)^2 - Æ^2}\,e^{2\theta}y = 0 \ldots(2).$$

3. The forms (1) and (2) are convertible, like those on p. 428 of Boole's *Differential Equations*, 2nd edition, 1865, by changing θ into $-\theta$; and the change of θ into $n\theta$ in either of them, would give the forms assumed by the equation when the independent variable is any power of $\cos x$ or of $\sec x$.

II. *Solution by the* Proposer.

Change the independent variable from x to arc tan x, then (1) is changed into

$$y'' + 2\frac{Ax^{-1} + (E+1)\,x}{1+x^2}\,y' + \frac{Æ^2 - (A-E)^2}{(1+x^2)^2}\,y = 0 \ldots\ldots(2).$$

Let $y = (1+x^2)^{-m} u$, then (2) becomes

$$u'' + 2\{Ax^{-1} + (E+1-2m)x\}\frac{u'}{1+x^2}$$

$$+\{Æ^2-(A-E)^2-2m(1+2A)+2m(2m-1-2E)x^2\}\frac{u}{(1+x^2)^2} = 0,$$

which will be binomial if

$$Æ^2-(A-E)^2-2m(2A+1) = 2m(2m-2E-1),$$

or $$(2m)^2+2(A-E)(2m) = Æ^2-(A-E)^2,$$

or $$Æ^2 = (A-E-2m)^2, \text{ or } 2m = -A+E\pm Æ.$$

Substituting for m, we get

$$(1+x^2)u'' + 2\{Ax^{-1} + (A+1\mp Æ)x\}u'$$

$$+\left[\left\{Æ\mp(A+\tfrac{1}{2})\right\}^2-(E+\tfrac{1}{2})^2\right]u = 0,$$

which is a binomial.

6339. (By Prof. SYLVESTER, F.R.S.)—Understanding by an Algebraical Perimeter of a rectilinear figure, the sum of the sides, each taken with the positive or negative sign; prove that the necessary and sufficient condition of all the curves described by points rigidly attached to the connecting link in a 3-bar link-work being unicursal is that one of the algebraical perimeters of the quadrilateral (real or impossible) formed by the three links and the line of centres shall be equal to zero.

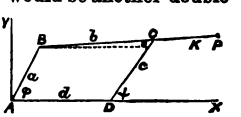
[Professor SYLVESTER states that ROBERTS' two cases of inverse conical 3-bar motion are special cases of unicursal 3-bar motion.]

Solution by SAMUEL ROBERTS, M.A.

Professor CAYLEY has shown (*Mathematical Society's Proceedings*, Vol. IV., pp. 109, 110) that the condition specified in the question is sufficient, and notices the cases of reduction of order.

To show the necessity of the condition, I consider that in general the curves described by points rigidly connected with the traversing bar, have a one-to-one correspondence, and have therefore the same deficiency. We may therefore regard a point on the traversing bar as describing the locus, and since the extremities of the links which rotate describe exceptional loci, we will suppose the describing point different from these.

The locus described is evidently symmetrical about the line of centres. Moreover, since the circular points are triple points, and there are three double points on the line of centres, we must, in order to have a unicursal curve, have on this line a contact of two branches whose common tangent is perpendicular to the line of centres. For we cannot have four double points on the line of centres identically, nor one double point finitely distant from that line; since, then, there would be another double point on the other side of it. Now, taking A as origin, and the line of centres as the axis of x (see figure), we have



$$x = a\cos\phi + (b+K)\cos\theta,$$

$$y = a\sin\phi + (b+K)\sin\theta,$$

$$x = c\cos\psi + K\cos\theta + d, \quad y = c\sin\psi + K\sin\theta;$$

and the equation of the locus is obtained from these in the form

$$y^2\{K[x^2+y^2+(b+K)^2-a^2]-(b+K)[(x-d)^2+y^2+K^2-c^2]\}^2$$
$$+\{K(x-d)[x^2+y^2+(b+K)^2-a^2]-(b+K)x[(x-d)^2+y^2+K^2-c^2]\}^2$$
$$=4d^2K^2(b+K)^2y^2.$$

The three double points on the axis of x are determined by

$$K(x-d)\{x^2+(b+K)^2-a^2\}-(b+K)x\{(x-d)^2+K^2-c^2\}=0\ldots(a);$$

and since the coefficient of y^2 must vanish, for x = root of (a), we have also

$$K\{x^2+(b+K)^2-a^2\}-(b+K)\{(x-d)^2+K^2-c^2\}=0\ldots\ldots(b),$$
$$\pm 2dK(b+K).$$

Multiplying (b) by x, and subtracting (a) from the result, we have

$$Kd\{x^2+(b+K)^2-a^2\}\pm 2dK(b+K)x=0,$$

or

$$Kd\{[x\pm(b+K)]^2-a^2\}=0, \text{ or } x=\mp(b+K)\pm a.$$

Substituting this value of x in (b), and remarking that $-(b+K)$ corresponds to $+2dK(b+K)x$, and $+(b+K)$ to $-2dK(b+K)x$, we get the conditions $(b+d\mp a)^2-c^2=0$ or $(b-d\pm a)-c^2=0$. We disregard the factors K, $b+K$, d as irrelevant and corresponding to the excluded extremities of the rotating links or to an evanescent bar.

6384. (By Dr. HOPKINSON, F.R.S.)—A heavy wire of uniform section is carried on a series of supports in the same horizontal plane, L_r being the bending moment at the r^{th} point of support, l_r the distance between the $(r-1)^{th}$ and r^{th} support, and m the mass of the wire per unit length; prove that $L_{r-1}l_r+2L_r(l_r+l_{r-1})+L_{r+1}l_{r+1}=\frac{1}{4}mg(l_r^3+l_{r+1}^3)$.

Solution by Professor MINCHIN, M.A.

This is the famous "Equation of Three Moments," restricted to the case of a uniformly loaded continuous beam. Demonstrations of the equation will be found in various works on Applied Mechanics; among others, see those of BRESSE and COLLIGNON.

Briefly the proof is this,—Let A, B, C be any three consecutive points of support; let M_1, M_2, M_3 be the bending moments at these points; A_0 and A_1 the shearing forces just behind and just in front of A; B_0 and B_1, C_0 and C_1 the shearing forces just behind and just in front of B and C respectively; l = AB, l' = BC; w and w' the loads per unit length in the spans AB and BC.

Now the bending moment at any section is equal to $\frac{EI}{\rho}$, in the well-known notation of the subject. First take B as origin, BA as axis of x, and the downward vertical as axis of y. Then, if x, y are the coordinates of any point of the beam between B and A,

$$EI\frac{d^2y}{dx^2}=M_2-B_0x+\tfrac{1}{2}wx^2,$$

therefore $EIy = EI \tan \alpha \cdot x + \frac{1}{2}M_2x^2 - \frac{1}{6}B_0x^3 + \frac{1}{24}wx^4$ .....(1),
where α = inclination to BA of tangent to the beam at B.

Again, taking the upward vertical at B as axis of y, and BC as axis of x, we have $\frac{1}{\rho} = -\frac{d^2y}{dx^2}$ (nearly) at any point between B and C, so that

$$-EI\frac{d^2y}{dx^2} = M_2 - B_1x + \frac{1}{2}w'x^2,$$

therefore $-EIy = -EI \tan \alpha \cdot x + \frac{1}{2}M_2x^2 - \frac{1}{6}B_1x^3 + \frac{1}{24}w'x^4$.....(2).
Now, when $x = l$, $y = 0$ in (1), and when $x = l'$, $y = 0$ in (2). Express this, and add, so as to get rid of tan α, and we have

$$\frac{1}{2}(l+l')M_2 - \frac{1}{6}(B_0l^2 + B_1l'^2) + \frac{1}{24}(wl^3 + w'l'^3) = 0 \text{ .....(3).}$$

Also, for the equilibrium of the span AB, we have, by moments about A,

$$lB_0 = M_2 - M_1 + \frac{1}{2}wl^2 \text{.....(4),}$$

and for the equilibrium of the span BC, by moments about C,

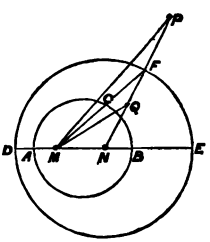
$$l'B_1 = M_2 - M_1 + \frac{1}{2}w'l'^2 \text{ .....(5).}$$

Into (3) put the values of B_0 and B_1 from (4) and (5), and we get the general "Equation of Three Moments,"

$$8(l+l')M_2 + 4(M_1l + M_2l') = wl^3 + w'l'^3.$$

6410 (By Prof. Genese, M.A.)—Conics are drawn with a common focus, and constant major axis: if their centres lie on a fixed circle, prove that the envelop of the series will be a conic having the same focus.

Solution by Professor Townsend, F.R.S.



Let M be the common focus of the several conics, ABC the circle locus of their several centres, DEF the associated circle locus of their several second foci, C and F the centre and second focus of any conic of the system, C' and F' (to be supplied) those of an infinitely near consecutive conic in either direction, P and Q the two points of intersection of the two conics, a the constant semi-axis of the system, and b the radius of the circle locus of their several centres; then since, by hypothesis, $PM \pm PF' = PM \pm PF' = 2a$, and $QM \pm QF = QM \pm QF' = 2a$, therefore $PF' = PF$ and $QF' = QF$, and therefore P and Q both lie on the radius FN at F to the circle DEF. Hence since, for the point P external to the radius, $PM \pm PF = 2a$, and $PN - PF = 2b$, therefore for that point $PM \pm PN = 2(a \pm b)$; and since, for the point Q internal to the radius, $QM \pm QF = 2a$, and $QN + QF = 2b$, therefore for that point $QM \mp QN = 2(a \mp b)$. The complete envelop of the system of conics consists, therefore, of an ellipse and of an hyperbola, of which alike M and N are the foci, and of which respectively $a+b$ and $a-b$ are the semi-axes.

6375. (By Prof. Crofton, F.R.S.)—If a heavy particle be moving, up or down, on a curve (rough or smooth), show that, if it leaves the curve, the parabola described has contact of the second order with the curve.

Solution by C. J. Monro, M.A.; G. F. Walker, M.A.; *and others.*

The particle will quit the curve on the concave (or convex) side as soon as the curvature of the orbit which it would describe if the curve had just come to an end has become greater (or less) than the curvature of the curve, and friction will only promote or prevent the satisfying of the condition. In either case when the particle quits the curve the curvature of its free orbit will be equal to that of the curve; and this is another way of stating the proposed theorem. The free orbit need not be a parabola, for the above holds good whatever the forces. *E.g.*, if there are none, the particle will quit the curve at a point of inflexion or a cusp only.

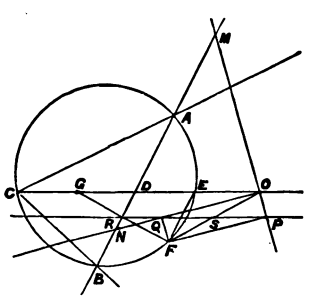
6245. (By Prof. Cochez.)—Démontrer que l'enveloppe des axes des coniques tangentes à deux droites en deux points donnés est une parabole.

I. *Solution by* R. F. Davis, M.A.; Prof. Genese, M.A.; *and others.*

Let OP, OP′ be two tangents to a conic; PG, P′G′ the corresponding normals measured to the axis; and CD, CD′ the corresponding semi-conjugate diameters. Then OP : OP′ = CD : CD′, and PG : CD :: $b : a$ :: P′G′ : CD′; wherefore OP : OP′ :: PG : P′G′. Now P, P′ are fixed points, and PG, P′G′ fixed directions; the envelop therefore of the axis GG′ is a parabola touching the two normals and the chord of contact (*Reprint*, XXII., p. 85). It may be noticed that the orthocentres of the triangles formed by any three of the five straight lines—the chord of contact, the two normals, and the two axes—all lie upon the directrix CO.

II. *Solution by* Prof. Townsend, F.R.S.; Prof. Matz, M.A.; *and others.*

If, in the figure, AC and BC be the two right lines, touched at A and B by all the conics of the system, D the middle point of the common chord of contact AB, E the second point at which the line of centres CD of the system is met by the circle ABC, and EF the chord of ABC through E parallel to AB; the parabola of which F is the focus and CD the directrix is that of the question.



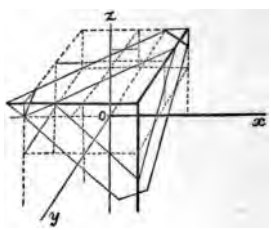
For, if O, on the line of centres CD, be the centre of any conic of the system, M and N the two points at which any pair of its conjugate diameters OM and ON intersect the chord of contact AB, and G the reflexion of F with respect to the axis AB, which point lies manifestly on

the line of centres CD at the distance from D equal and opposite to that of E; then, since, by a familiar property of the circle, DE, and therefore DG, is the third proportional to CD and AD or BD, and since consequently, by a fundamental property of central conics, the rectangle DM . DN is equal to the rectangle DG . DO, the four points G, M, N, O lie always on a circle, which, when OM and ON are at right angles to each other, and therefore the axes of the conic, has its centre on AB, and passes consequently through F as well as through G; hence, the feet P and Q of the perpendiculars FP and FQ from F upon the axes OM and ON of the conic are collinear at once with the middle points R and S of the two rectilinear segments FG and FO, and lie consequently on the fixed right line RS parallel to CD, and lying midway between it and F; and as consequently, of the two variable right angles FPM and FQN, two of the sides FP and FQ turn round the fixed point F, while the two vertices P and Q move on the fixed right line RS, therefore the two remaining sides PM and QN envelop the parabola of which F is the focus and RS the tangent at the vertex, that is, the parabola of which F is the focus and CD the directrix; and therefore, &c.

6382. (By ELIZABETH BLACKWOOD.)—If x, y, z be each taken at random between a and $-a$, what is the chance that $x+y+z$ will be between b and $-b$? Explain any easy experimental method by which your result may be verified. [Miss BLACKWOOD states that, taking a and b each $=1$, she has found the event to happen 65 times in 100 trials.]

Solution by G. F. WALKER, M.A.; D. EASTWOOD, M.A.; *and others.*

Consider the section of the sides of the cube $x = \pm a$, $y = \pm a$, $z = \pm a$ by the two planes $x+y+z = \pm b$.



(1) Let $b > a$. The planes will cut off from the cube two tetrahedrons whose edges are $3a-b$, and the combined volume of these is $2 \cdot \frac{1}{6}(3a-b)^2$, and taking xyz as the coordinates of a point, and all points in the cube being equally likely, the chance is the ratio of the volume of the cube between the planes and the whole cube

$$= \frac{(2a)^3 - 2 \cdot \frac{1}{6}(3a-b)^3}{(2a)^3}$$

$$= \frac{3(2a)^3 - 2(3a-b)^3}{24a^3} = \frac{b^3 - 9ab^2 + 27a^2b - 3a^3}{24a^3}.$$

(2) Let $b < a$. The planes now cut the cube in hexagons, and the volume now cut off by one of them is

$$\tfrac{1}{6}[(3a-b)^3 - 3(a-b)^3],$$

and chance

$$= \frac{(2a)^3 - \frac{1}{6}[(3a-b)^3 - 3(a-b)^3]}{(2a)^3}$$

$$= \frac{24a^3 - [24a^3 - 18a^2b + 2b^3]}{3(2a)^3} = \frac{18a^2b - 2b^3}{24a^3}.$$

If $a = b$, both of these become $\frac{2}{3}$, which is very near to $\frac{65}{100}$.

6353. (By the Rev. J. VENN, M.A.)—All x is either y and not z, or z and not y. All wx is either both y and z or neither of them. All xy which is not w is z. No x that is neither y nor w is z. Show that under these conditions there is no x.

Solution by HUGH MCCOLL, B.A.

Speaking throughout of some *one* originally unclassed individual, let x, y, z respectively denote the statements,—it belongs to the class x, it belongs to the class y, it belongs to the class z. Then the data will be expressed by the complex statement

$$(x : yz' + zy')\,(wx : yz + y'z')\,(xyw' : z)\,(xy'w' : z).$$

Let $f(x)$ denote this statement. We have to show that $xf(x) : 0$, that is, that $f(x) : x'$. Now $xf(x) : f(1)$, and since every implication $\alpha : \beta$ has a disjunction consequent $\alpha' + \beta$, we get

$$f(1) : (yz' + zy')\,(w' + \overline{yz + y'z'})\,(y' + \overline{w} + z)\,(y + \overline{w} + z').$$

We over-line $yz + y'z'$ as zero, because it is the denial of the first factor $yz' + zy'$; we then over-line w as the denial of the (then) second factor w'; the product of the last two factors is $yz + y'z'$, which is the denial of $yz' + zw'$. We thus find (without the trouble of multiplication) that $xf(x) : f(1) : 0$, that is, that $f(x) : x'$.

6357. (By J. E. A. STEGGALL, M.A.)—Find in how many ways a given number n may be split up into the sum of 1, 2, 3 ... different numbers. *Example.*—In how many ways can 34 be split up into 4 different numbers?

Solution by W. A. WHITWORTH, M.A.; Prof. GENESE, M.A.; *and others.*

Let P_r^n denote the number of r-partitions of n; then we have to cast out from P_4^{34} all the partitions in which the four numbers are not all "different." But those in which there are two *ones*, two *twos*, two *threes*, are P_2^{32}, P_2^{30}, P_2^{28}, and so on. Hence the total number of doublets among the partitions is $P_2^{32} + P_2^{30} + P_2^{28} + \ldots + P_2^2$.

But, if we were to reject all these, we should be rejecting *twice* every partition which consisted of two doublets. The number of such partitions is evidently P_2^{17}. Replacing these, we get the final result

$$P_4^{34} - \{P_2^{32} + P_2^{30} + \ldots + P_2^2\} + P_2^{17}.$$

Now (*Choice and Chance*, p. 94), $P_4^{34} = \dfrac{34^3 + 3 \cdot 34^2 - 4}{144} = 297$; and P_2^{2r} or $P_2^{2r+1} = r$; hence the required number is

$$297 - (16 + 15 + 14 + \ldots + 1) + 8 = 169.$$

6358. (By R. Tucker, M.A.)—If O, P are the equidistant angle-point and the orthocentre of a triangle, and ρ_1, ρ_2, ρ_3 the radii of circles about OPA, OPB, OPC respectively; show (1) that

$$OP^2(\rho_1^{-2}+\rho_2^{-2}+\rho_3^{-2}) = 8[1-\cos(A-B)\cos(B-C)\cos(C-A)];$$

and (2) that, if any angle, as C, = 60°, then four of the points, as A, B, O, P, are concyclic. [The terms "equidistant angle-point," and "equidistant side-point," Mr. Tucker proposes to use in place of "centre of circumscribed circle" and "centre of inscribed circle" of a triangle.]

Solution by G. Heppel, M.A.; A. McMurchy, B.A.; *and others.*

Here we have $\angle OAP = B-C$, therefore $2\rho_1 \sin(B-C) = OP$, whence

$$OP^2(\rho_1^{-2}+\rho_2^{-2}+\rho_3^{-2}) = 4[\sin^2(B-C)+\sin^2(C-A)+\sin^2(A-B)]$$
$$= 8[1-\cos(A-B)\cos(B-C)\cos(C-A)].$$

If $C=60°$, $A+B=2c$, therefore $B-C=C-A$, therefore $\angle OAP = \angle OBP$, and A, B, O, P are concyclic.

6340. (By Prof. Townsend, F.R.S.)—The normal at a variable point on a fixed surface being supposed to intersect in every position a fixed right line in the space of the surface; prove, from the requisite equation of condition, that—

(*a*) Whatever be the position of the line in the space, the curve locus of the variable point is the intersection with the surface of another of the same order, passing through its several points of intersection with the line, and through the several points at which its tangent planes are perpendicular to the line.

(*b*) When the line at any of the aforesaid points of intersection is itself a normal to the surface, its point of normal intersection is a crunodal double point of the aforesaid curve, the two tangents at which intersect at right angles, and coincide in direction with the two principal tangents to the surface at the point.

I. *Solution by* C. J. Monro, M.A.; G. Eastwood, M.A.; *and others.*

Let $U = 0$ determine the surface, and take the right line for z-axis. One of the equations to the normal gives the condition

$$\frac{dU}{dy}x - \frac{dU}{dx}y = 0 \quad \ldots\ldots(1);$$

whence (*a*) follows immediately.

(*b*) Measure x and y along principal tangents; then U is of the form $cz + ax^2 + by^2 + T$, and (1) becomes $xy + T' = 0$, where T and T′ are of the same order of magnitude as x^3, y^3, z^3, at most. These give a curve touching the principal tangents where they cross.

II. *Solution by the* PROPOSER; J. L. MCKENZIE, B.A.; *and others.*

To prove (*a*).—If $\phi = 0$ be the equation of the surface, referred to any rectangular triad of coordinate planes intersecting at any origin on the fixed line in the space, x, y, z the coordinates of the variable points in any position on the surface, and l, m, n those of any fixed points on the line; then since, by hypothesis, the line through the origin, the cosines of whose direction angles are proportional to $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, must, in every position of xyz on the surface, be complanar with the two the cosines of whose direction angles are proportional to x, y, z and to l, m, n, and since consequently the determinant of the nine cosines must vanish, we have, consequently, between the coordinates x, y, z, the equation

$$(mz-ny)\frac{d\phi}{dx} + (nx-lz)\frac{d\phi}{dy} + (ly-mx)\frac{d\phi}{dz} = 0;$$

which being of the same order with ϕ, and being satisfied by the two systems of values of x, y, z for which, respectively,

$$x : y : z = l : m : n \quad \text{and} \quad \frac{d\phi}{dx} : \frac{d\phi}{dy} : \frac{d\phi}{dz} = l : m : n;$$

therefore &c., as regards (*a*).

To prove (*b*).—Taking, in that case, the line itself as axis of z, and the tangent plane at its point of normal intersection with the surface as plane of xy, so that, consequently,

$$\phi = mz + \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \&c.;$$

then, since in the above equation $l = 0$ and $m = 0$, the equation becomes reduced to

$$y\frac{d\phi}{dx} - x\frac{d\phi}{dy} = 0;$$

or, from the value of ϕ in this case, to

$$h(y^2-x^2) + (a-b)xy + \&c. = 0,$$

which shows that the origin is a connodal double point on the curve in question, the tangents at which

$$h(y^2-x^2) + (a-b)xy = 0$$

are at right angles to each other, and bisect internally and externally the angle

$$ax^2 + 2hxy + by^2 = 0$$

determined by the two inflexional tangents to the surface at the point; and therefore &c., as regards (*b*).

N.B.—The known directions of the two "lines of curvature" through any point of a surface of any order are obviously given immediately by this latter property.

6396. (By C. LEUDESDORF, M.A.)—Show that the square of the tangent from any point P to the circle inscribed in the triangle ABC is

$$\frac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - r(r+2R),$$

r being the radius of this circle, and R that of the circumscribed circle.

Solution by G. TURRIFF, M.A.; E. W. SYMONS, M.A.; *and others.*

If O be the centre of the inscribed circle, then (TOWNSEND's *Mod. Geom.*, Art. 102) $OP^2 = \frac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - \frac{abc}{a+b+c}$,

$$2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{a+b+c}, \quad (\text{Tangent})^2 = OP^2 - r^2 = \&c.$$

[It is remarked that Quest. 6392 may be proved in a similar way.]

5341. (By J. W. SHARPE, M.A.)—Prove that, if $\alpha, \beta, \gamma, \rho$ be any four vectors, then $V\rho\alpha \cdot V\beta\gamma + V\rho\beta \cdot V\gamma\alpha + V\rho\gamma \cdot V\alpha\beta$ is also a vector.

Solution by CHRISTINE LADD; *the* PROPOSER; *and others.*

$$\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3S\alpha\beta\gamma;$$

for the vector of the left side is zero, since it is

$$\gamma S\alpha\beta - \beta S\gamma\alpha + \alpha S\beta\gamma - \gamma S\alpha\beta + \beta S\gamma\alpha - \alpha S\beta\gamma;$$

and the scalars of the two sides are clearly the same. Operate upon both sides with $S\rho$, and we get $S(\rho\alpha V\beta\gamma + \rho\beta V\gamma\alpha + \rho\gamma V\alpha\beta) = 0$;

hence $S(V\rho\alpha \cdot V\beta\gamma + V\rho\beta \cdot V\gamma\alpha + V\rho\gamma \cdot V\alpha\beta) = 0$;

hence $V\beta\gamma \cdot V\alpha\rho + V\gamma\alpha - V\beta\rho + V\alpha\beta \cdot V\gamma\rho$ denotes a vector.

6320. (By ELIZABETH BLACKWOOD.)—Given that $ax + by$ is less than c and greater than d, and that $cx + dy$ is less than a and greater than b; required the limits of x and y, a, b, c, d being all positive.

Solution by Prof. TANNER, M.A.; J. A. KEALY, M.A.; *and others.*

We have $ax + by = (c-d)t + d$, $cx + dy = (a-b)t' + b$; where t, t' are positive proper fractions; hence

$$(ad-bc)x = (c-d)dt + d^2 - (a-b)bt' - b^2.$$

Take $ad-bc$ to be positive; then the greatest value of x is given by $t=1$, $t'=0$, the least by $t=0$, $t'=1$; hence

$$\frac{cd-b^2}{ad-bc} > x > \frac{d^2-ab}{ad-bc}; \text{ so, too, } \frac{a^2-cd}{ad-bc} > y > \frac{ab-c^2}{ad-bc}.$$

If $ad-bc$ be negative, the x limits are transposed, and likewise those for y. If $ad-bc=0$, the data are contradictory, unless $a, b = c, d$, when they are insufficient.

5624. (By Professor Sylvester, F.R.S.)—If X_j is to be a rational homogeneous function of the j^{th} order of i independent quantities $a, b, c, \ldots k, l$, and is to satisfy the partial differential equation

$$(A\delta_a + B\delta_b + \ldots + L\delta_l)\, X_j = 0,$$

where A, B, ... L are homogeneous linear functions of $a, b, \ldots, l$, not absolutely independent, but connected by a single homogeneous linear equation with one another; prove that the number of arbitrary parameters in X_j will in general be the number of solutions in positive integers (zero counted in as positive) of the simultaneous equations

$$u_0 + u_1 + u_2 + \ldots + u_{i-1} = j\tfrac{1}{2}v_0 + u_1 + 2u_2 + \ldots (i-1)\, u_{i-1} = (i-1)\tfrac{1}{2}jv_0 + v_1 = 1.$$

Or, again, suppose $[(a-b)\,\delta_c + (b-c)\,\delta_a + (c-a)\,\delta_b]\, X_2 = 0$, the complete value of X_2, viz., $k\,(a^2+b^2+c^2) + k'(ab+ac+bc)$, contains 2 parameters k, k', which will be found to agree with the rule; and so, if in the above we write X_{22} or X_{2i+1} in place of X_2, it will be found (still in accordance with the rule) that there will be $i+1$ arbitrary parameters in each of these forms.

Example:—Let $i=2$, then, if $(\lambda A\delta_a - \mu A\delta_b)\, X_j = i$, $X_j = k\,(\mu a + \lambda b)^j$, and contains just *one* arbitrary parameter k, which agrees with the rule.

Solution by the Proposer.

This is a simple consequence of the rule—*re* Invariants—given in the *Philosophical Magazine*, some time ago, but is far from being obviously so.

By linear substitutions, the equation may be changed *in general* into

$$(A'\delta_{B'} + B'\delta_{C'} + \ldots + K'\delta_{L'})\, X'_j = 0,$$

where X′ is now regarded as a function of A′, B′, ... L′; and then, by some simple, but a trifle subtle argumentation, it may be shown that the number of arbitrary parameters required is the total number of invariants and covariants of the order j to a quartic of the $(i-1)$ degree, which by virtue of a known theorem is the number of solutions of the equations in the question. I think it rather valuable as carrying the invariantive theory into a higher sphere.

It may be observed that $v_0 + v_1 = 1$ simply says that v_0 is to be 0 or 1 according as $(i-1)j$ is even or odd.

It would probably be difficult to prove the theorem without having recourse to the theory of invariants, as one would never have thought otherwise of the transformation into $A'\delta_{B_1} + \ldots$, and even then it is not easy (without the light of this theory) to see that the general number of solutions might not exceed the number stated.

6188. (By Professor Sylvester, F.R.S.)—If in any persymmetrical matrix of the n^{th} order, each column in succession calling its elements $c_1, c_2 \ldots c_n$, be replaced by $0, c_1, 2c_2 \ldots (n-1)\, c_n$, show that the sum of the n determinants thus formed is zero. For example, show that

$$\begin{vmatrix} 0 & b & c & d \\ a & c & d & e \\ 2b & d & e & f \\ 3c & e & f & g \end{vmatrix} + \begin{vmatrix} a & 0 & c & d \\ b & b & d & e \\ c & 2c & e & f \\ d & 3d & f & g \end{vmatrix} + \begin{vmatrix} a & b & 0 & d \\ b & c & c & e \\ c & d & 2d & f \\ d & e & 3e & g \end{vmatrix} + \begin{vmatrix} a & b & c & 0 \\ b & c & d & d \\ c & d & e & 2e \\ d & e & f & 3f \end{vmatrix} = 0.$$

Solution by H. STABENOW, M.A.; J. HAMMOND, M.A.; *and others.*

a, b, c, &c. may be considered as s_0, s_1, s_2, &c., the sums of the powers of α, β, γ, &c., the roots of an equation. The persymmetrical determinant of the n^{th} order will then be (SALMON'S *Higher Algebra*, 3rd ed., p. 22)

$$\Sigma (\alpha-\beta)^2 (\beta-\gamma)^2 (\alpha-\gamma)^2 \ldots \text{ to } \tfrac{1}{2}n(n-1) \text{ factors.}$$

Now, operating on both sides with $\left(\frac{d}{da}+\frac{d}{d\beta}+\ldots\right) \equiv \Delta$, and observing that $\Delta s_r = r\Delta s_{r-1}$, the result follows at once, since

$$\Delta\Sigma (\alpha-\beta)^2 (\beta-\gamma)^2 (\alpha-\gamma)^2 \ldots = 0.$$

6207. (By E. ANTHONY, M.A.) — In a plane triangle ABC, if $\frac{1}{2}\Sigma \sin^2 A \equiv \sigma$, and $\sigma - \sin^2 A = \sigma_1$, &c., prove that

$$\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2 = \sin^2 A \sin^2 B \sin^2 C.$$

Solution by A. MCMURCHY, B.A.; K. KNOWLES, B.A., L.P.C.; *and others.*

Writing *sines* for sides in the well known zero-homogeneous expressions for cos A, we have $\sigma_1 = \sin B \sin C \cos A$;

therefore $\sigma_2\sigma_3 + \ldots + \ldots$

$= \sin A \sin B \sin C (\sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B)$

$= \sin^2 A \sin B \sin C (\cos A + \cos B \cos C)$

$= \sin^2 A \sin^2 B \sin^2 C.$

6216. (By R. TUCKER, M.A.) — O is the point of projection of two equi-horizontal-range particles, and (OP, OP'), (OQ, OQ') the normal chords and diameters of curvature at O respectively; prove that, if the initial velocity be the same for both, PQ' is parallel to QP'.

Solution by A. ANDERSON, M.A.; *the* PROPOSER; *and others.*

The equation to the path is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}, \text{ or } gr \cos^2 \theta = 2u^2 \cos \alpha \sin (\alpha-\theta) \quad \ldots(1).$$

Since OP is a normal chord ($\theta = \alpha - \frac{1}{2}\pi$), therefore

$$\text{OP} = \frac{2u^2}{g} \cot \alpha \operatorname{cosec} \alpha = 4h \cot \alpha \operatorname{cosec} \alpha,$$

$$\text{OP}' = \frac{2u^2}{g} \tan \alpha \sec \alpha = 4h \tan \alpha \sec \alpha.$$

Now OQ is the diameter of curvature at O, hence we have

$$OQ = \frac{4SO^2}{SY} = \frac{4h^2}{h\cos\alpha} = 4h\sec\alpha \text{ (SY perpendicular on tangent at O).}$$

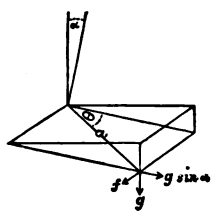
Similarly $OQ' = 4h\operatorname{cosec}\alpha$;

therefore $OP \cdot OP' = OQ \cdot OQ'$; hence PQ′ is parallel to QP′.

6259. (By G. S. Carr, B.A.)—A railway carriage door of width $2a$ and mass m, with the axis through its hinges inclined α to the vertical in a plane at right angles to the rails, hangs open at rest. The carriage suddenly moves forward with an initial velocity v, and a constant acceleration afterwards f. Show that—(1) if the door slams, the force of the blow on the jamb is $\frac{1}{4}m[v^2+\frac{8}{3}a(f-g\sin\alpha)]^{\frac{1}{2}}$; and (2) the time of a small oscillation of the door about the position of equilibrium, is

$$4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}(f^2+g^2\sin^2\alpha)^{-\frac{1}{4}}.$$

Solution by the Proposer.



Let u be the initial velocity of the centre of inertia of the door; w, w' its initial and final angular velocities; P, P′ the corresponding impulsive forces upon the hinges and the jamb. Then $P = mu$, $Pa = mk^2w$, and $u + aw = v$; therefore $w = va \div (a^2+k^2)$.....(1).

Reduce the axis of the door to rest by applying to the centre of inertia the reversed acceleration f and the reversed initial velocity v. The initial velocity of the centre of inertia will then become aw in the direction opposite to u. The equation of *vis viva* is now

$$m(a^2+k^2)(w'^2-w^2) = 2(mfa-mga\sin\alpha), \text{ and } k^2 = \tfrac{1}{3}a^2;$$

$$\therefore\ w' = \left\{w^2+\frac{3}{2a}(f-g\sin\alpha)\right\}^{\frac{1}{2}} = \frac{3}{4a}\left\{v^2+\frac{3}{8a}(f-g\sin\alpha)\right\}^{\frac{1}{2}}, \text{ by (1).}$$

By moments about the axis we have $2aP' = m(a^2+k^2)w'$;

therefore $P' = \frac{2}{3}maw' = \frac{1}{4}m\left\{v^2+\frac{8}{3}a(f-g\sin\alpha)\right\}^{\frac{1}{2}}$.

Secondly, to find the time of oscillation. The equation of motion is

$$(k^2+a^2)\ddot{\theta} = fa\cos\theta - g\sin\alpha\, a\sin\theta \text{.....(2).}$$

Put $\ddot{\theta} = 0$ to find the position of equilibrium; thus

$$f\cos\theta_0 - g\sin\alpha\sin\theta_0 = 0 \text{.....(3).}$$

For a small oscillation put $\theta = \theta_0 + \phi$ in (2); then we have

$$\tfrac{4}{3}a\ddot{\phi} = f(\cos\theta_0 - \phi\sin\theta_0) - g\sin\alpha(\sin\theta_0 + \phi\cos\theta_0)$$
$$= -\phi(f\sin\theta_0 + g\sin\alpha\cos\theta_0), \text{ by (3).}$$

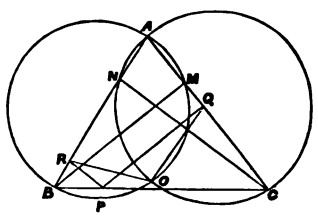
Eliminate θ_0 by (3); thus $\ddot{\phi} + \frac{3}{4}a\left\{f^2+g^2\sin^2\alpha\right\}^{\frac{1}{2}}\phi = 0$.

Therefore time of oscillation $= 4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}\left\{f^2+g^2\sin^2\alpha\right\}^{-\frac{1}{4}}$.

6365. (By H. Murphy.)—If from a variable point on any side of a triangle two lines are drawn making given angles with it; prove that the circle through the points of intersection with the other sides and opposite vertex passes also through a fourth fixed point.

Solution by Prof. Genese, M.A.; G. Heppel, M.A.; *and others.*

Let PQ, PR be two of the lines in question, and let BM, CN be parallels thereto; also let the circles ABM, ACN intersect in O; then the triangles OCM, ONB are similar, and CQ : QM = CP : BP = NR : RB; hence the triangles OCQ and ONR are similar; therefore ∠NOR = COQ, and ∠QOR = CON = supplement of A. Thus the circle AQR passes through O.



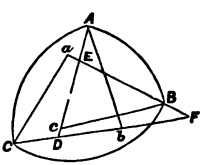
6214. (By G. Heppel, M.A.)—If the sides of a plane triangle be circular arcs of equal radius, find a simple necessary relation between the sides and angles of the triangles.

Solution by J. J. Walker, M.A.

Let a, b, c be the centres of the equiradial arcs BC, CA, AB, and also stand for the circular measures of those arcs respectively. Suppose Ac and Cb to meet in D; Cb and Ba in E; Ba and Ac in E. Then, if all the sides are *concave* towards the opposite corners,

$A - b = \angle EDF$, $B - c = \angle DEF$, $C - A = \angle DFE$,

so that $A + B + C - (a + b + c) = \pi$. But if any side is *convex* towards its opposite corner, it must be taken as negative in the above general relation, as appears by considering that, if, *e.g.*, BC became convex towards A, each of the angles B, C would be diminished by a. When the radius becomes infinite, the lengths of BC, CA, AB remaining finite, $A + B + C = \pi$.

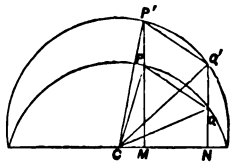


6151. (By G. Turriff, M.A.)—If P, Q be two points on an ellipse such that the difference (α) of their excentric angles is constant, and if the ordinates at P and Q be produced to meet the auxiliary circle in P′ and Q′; show (1) that area of $\Delta P'CQ'$: area of $\Delta PCQ = a : b$, C being the centre; (2) that PQ touches at its middle point the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \frac{1}{2}\alpha$.

Solution by C. MORGAN, B.A.; J. A. KEALY, M.A.; *and others.*

1. The ellipse is the projection of the auxiliary circle; hence, the triangle PCQ being the projection of P′CQ′, we have

$\Delta PCQ : \Delta CP'Q' = \cos.\text{projective } \angle = b : c.$



2. Let the excentric angles of P, Q be ϕ, $(\phi + \alpha)$, where α is constant; then the coordinates of P, Q are

$[a\cos\phi, b\sin\phi]$, $[a\cos(\phi+\alpha), b\sin(\phi+\alpha)]$;

and the coordinates of the middle point of PQ are

$$x' = a\cos(\phi + \tfrac{1}{2}\alpha)\cos\tfrac{1}{2}\alpha, \quad y' = b\sin(\phi + \tfrac{1}{2}\alpha)\cos\tfrac{1}{2}\alpha;$$

therefore $a^2y'^2 + b^2x'^2 = a^2b^2\cos^2\frac{1}{2}\alpha$, *i.e.* (x', y') is on the specified ellipse.

Again, the equation to the line PQ is

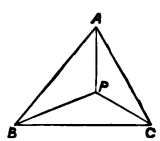
$$ay\sin(\phi + \tfrac{1}{2}\alpha) + bx\cos(\phi + \tfrac{1}{2}\alpha) = ab\cos\tfrac{1}{2}\alpha, \text{ or } a^2y'y + b^2x'x = a^2b^2\cos^2\tfrac{1}{2}\alpha,$$

which is a tangent to the ellipse specified in the Question.

6399. (By the Rev. H. G. DAY, M.A.)—If (u, v, w) are the distances of a point from the corners A, B, C of a triangle, and (l, m, n) are constants; find the point within the triangle such that $lu + mv + nw$ shall be a minimum, when a minimum is possible.

Solution by Prof. GENESE, M.A.; W. B. GROVE, B.A.; *and others.*

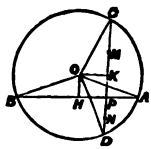
Let P be the required position; then, evidently, if the point P were acted on by three tensions towards A, B, C, proportional to l, m, n, these tensions would mutually balance; hence no minimum is possible unless lines proportional to l, m, n would form a triangle, and the three angles BPC, CPA, APB taken in order will be equal to the exterior angles of this triangle. The only other requisite condition is that BPC may be greater than A, &c., or that the sum of any two corresponding angles must be less than two right angles.



All these conditions can be derived from the obvious fact, that a circle with A as centre must touch in P a Cartesian oval $mv + nw =$ constant.

5947. (By Prof. SEITZ, M.A.)—Two points are taken at random within a circle, one on each side of a chord which divides the circle into two segments whose areas are u_1, u_2; show that the chance that the chord drawn through the points is less than a chord which cuts an area u from the circle, is $\frac{u^2}{u_1u_2}$, when $u < u_1 < u_2$, and $\frac{2u-u_1}{u_2}$, when $u > u_1$.

Solution by the PROPOSER.



Let M, N be two random points on opposite sides of the chord AB, CD the chord through them, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $PM = x$, $MN = y$, $PC = w_1$, $PD = w_2$, $\angle AOH = \alpha$, $\angle COK = \theta$, $\angle HOK = \phi$, and $2\beta =$ the angle formed by radii drawn to the extremities of a chord which cuts off an area, u.

Then we have $w_1 = r(\sin\theta + \cos\alpha \operatorname{cosec}\phi - \cos\theta\cot\phi)$,

$$w_2 = r(\sin\theta - \cos\alpha\operatorname{cosec}\phi + \cos\theta\cot\phi),$$

$$u = r^2(\beta - \sin\beta\cos\beta), \quad \text{and} \quad u_1 = r^2(\alpha - \sin\alpha\cos\alpha).$$

An element of the circle at M is $r\sin\theta\, d\theta\, dx$, and at N it is $d\phi\, y\, dy$.

When $u < u_1$, the limits of θ are 0 and β; of ϕ, $\alpha-\theta$ and $\alpha+\theta$, and doubled; of x, 0 and w_1; and of y, x and w_2+x. Hence, since the whole number of ways the two points can be taken is u_1u_2, we have

$$p = \frac{2}{u_1u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta}\int_0^{w_1}\int_x^{w_2+x} r\sin\theta\, d\theta\, d\phi\, dx\, y\, dy$$

$$= \frac{r}{u_1u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta}\int_0^{w_1}[(w_2+x)^2 - x^2]\sin\theta\, d\theta\, d\phi\, dx$$

$$= \frac{2r^2}{u_1u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta} w_1w_2\sin^2\theta\, d\theta\, d\phi = \frac{4r^4}{u_1u_2}\int_0^\beta(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta = \frac{u^2}{u_1u_2}.$$

When $u > u_1$, the limits of ϕ are $\alpha-\theta$ and $\alpha+\theta$, when $\theta < \alpha$; and $\theta-\alpha$ and $\theta+\alpha$, when $\theta > \alpha$. Hence we have

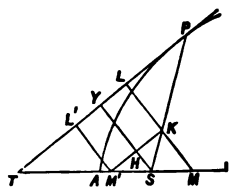
$$p_1 = \frac{2r^2}{u_1u_2}\int_0^\alpha\int_{\alpha-\theta}^{\alpha+\theta} w_1w_2\sin^2\theta\, d\theta\, d\phi + \frac{2r^2}{u_1u_2}\int_\alpha^\beta\int_{\theta-\alpha}^{\theta+\alpha} w_1w_2\sin^2\theta\, d\theta\, d\phi$$

$$= \frac{4r^4}{u_1u_2}\int_0^\alpha(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta + \frac{4r^4}{u_1u_2}\int_\alpha^\beta(\alpha - \sin\alpha\cos\alpha)\sin^2\theta\, d\theta = \frac{2u-u_1}{u_2}.$$

[When $u_1 = u_2 = \frac{1}{2}\pi r^2$, we have $p = \frac{1}{\pi^2}(2\beta - \sin 2\beta)^2$, which is the chance required in Prof. SYLVESTER'S Quest. 1849, whereof two other solutions are given on pp. 92, 93 of Vol. VIII. of our *Reprints*.]

6211. (By R. E. RILEY, B.A.)—Two points are taken on the axis of a parabola at equal distances on opposite sides of the focus, and from these points perpendiculars are drawn to a tangent. If SY be the perpendicular from the focus, show that the difference of the other two perpendiculars varies inversely as SY.

Solution by A. ANDERSON; C. MORGAN, B.A.; J. O'REGAN; *and others.*



Let ML, M'L' be the perpendiculars, and let SM = SM' = c; then we have

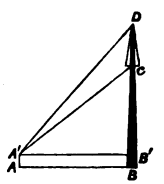
$$ML - M'L = 2c\left(\frac{SY}{ST}\right) = \frac{2c \cdot SY}{SP} \propto \frac{1}{SY},$$

since $SY^2 = SP \cdot SA$.

6315. (By Prof. MATZ, M.A.)—The dome of the Capitol at Washington rises m (= 396¼ feet) above the level of the ground, and is surmounted by the Statue of Freedom, whose height is n (= 18 feet); show that an observer whose eye is k (= 5¼ feet) above the ground, can get the best view of the Statue when his distance from the axis of the dome is

$$\{(m-k)[(m-k)+n]\}^{\frac{1}{2}} = 399{\cdot}8987372 + \text{feet}.$$

Solution by J. A. KEALY, M.A.; E. RUTTER; *and others.*



It is clear that, if a circle be drawn through the highest and lowest points (D, C) of the Statue, so as to touch at A' the line (A'B') through the observer's eye, parallel to the horizon, the visual angle CA'D will afford the observer the best view of the Statue; hence $A'B' = (B'C \cdot B'D)^{\frac{1}{2}} = 399{\cdot}8988425$ feet.

5889. (By J. J. WALKER, M.A.)—Show that $la + m\beta + n\gamma$ will be a side of the cone having origin as vertex, and the circle through the terms of α, β, γ as base, if

$$l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha) + n^{-1}T^2(\alpha-\beta) = 0;$$

and hence deduce immediately the existence of the second system of circular sections parallel to the plane through the terms of α^{-1}, β^{-1}, γ^{-1}.

Solution by G. TURRIFF, M.A.; Prof. EVANS; *and others.*

Let $S\epsilon\rho = -\epsilon^2$ be the equation of the plane through terms of α, β, γ, and $\rho^2 = 2S\delta\rho$ the equation of the sphere through the origin and terms of α, β, γ; hence the equation of the cone is $\rho^2\epsilon^2 + 2S\epsilon\rho S\delta\rho = 0$. The former two equations are satisfied by α, β, γ. Now, if $la + m\beta + n\gamma$ be a line in cone $(la + m\beta + n\gamma)^2\epsilon^2 + 2S\epsilon\,(la + m\beta + n\gamma)\,S\delta\,(la + m\beta + n\gamma) = 0$,

or $2lm\,S\alpha\beta + 2ln\,S\gamma\alpha + 2mn\,S\beta\gamma - (lm+ln)\,\alpha^2 - (ml+mn)\,\beta^2 - (nl+nm)\,\gamma^2 = 0.$

But $lm\,(\alpha-\beta)^2 = lm\alpha^2 + lm\beta^2 - 2lm\,S\alpha\beta$, &c., &c.;

therefore $lm\,(\alpha-\beta)^2 + mn\,(\beta-\gamma)^2 + nl\,(\gamma-\alpha)^2 = 0,$

or $n^{-1}T^2(\alpha-\beta) + l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha) = 0.$

Again, $l\alpha + m\beta + n\gamma$ may be written $l\alpha^2.\alpha^{-1} + m\beta^2.\beta^{-1} + n\gamma^2.\gamma^{-1}$, and taking $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ for (α, β, γ); we should get, in a similar way,

$(n\gamma^2)^{-1}T^2(\alpha^{-1}-\beta^{-1}) + (l\alpha^2)^{-1}T^2(\beta^{-1}-\gamma^{-1}) + (m\beta^2)^{-1}T^2(\gamma^{-1}-\alpha^{-1}) = 0,$

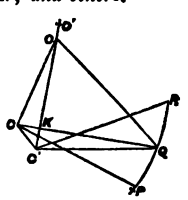
or $$\frac{n^{-1}T^2(\alpha-\beta) + l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha)}{\alpha^2\beta^2\gamma^2} = 0,$$

the same equation as the last; hence, &c.

6212. (By D. Edwardes.)—If ρ, R be the radii of absolute and spherical curvature at an ordinary point on a curve in space, and if ds, ds' be the elements of arcs at corresponding points on the locus of the centres of absolute and spherical curvature respectively, prove that

$$ds : ds' = d\rho : d\mathrm{R}.$$

Solution by G. J. Griffiths, M.A.; J. O'Regan; *and others.*



Let P, Q, R be three consecutive points on a curve; C, C′ the centres of absolute curvature at Q, R; O the centre of spherical curvature—the intersection of the consecutive normal planes OCQ, OC′R; $CQ = \rho$, $OQ = R$, $\angle CQC'$, the angle of torsion, $= dT = \angle OCO'$. Since C, C′ lie on the circle of which OQ is the diameter, $CC' \equiv ds = R\,dT$.

If O′ be the consecutive centre of spherical curvature, we may regard OO′ as an element of a plane curve, to which CO, C′O′ are consecutive tangents, and QC, QC′ perpendiculars on them from the origin Q. Remembering the expression $r\frac{dr}{dp}$, for the radius of curvature, we get

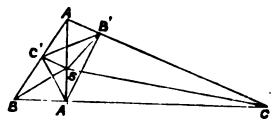
$$\frac{OO'}{\angle COC'} = R\frac{dR}{d\rho}, \text{ therefore } ds' = R\,dT\,\frac{dR}{d\rho};$$

whence $$ds : ds' = d\rho : d\mathrm{R}.$$

5620. (By E. W. Symons, M.A.)—Prove that the latus rectum of a conic inscribed in a triangle ABC is equal to $8R\frac{\delta_1\delta_2\delta_3}{d_1d_2d_3}$, where $\delta_1, \delta_2, \delta_3, d_1, d_2, d_3$ are the distances of the sides and vertices from either focus.

Solution by D. WICKERSHAM, B.D.; *the* PROPOSER; *and others.*

Let S be either focus; join SA, SB, SC; and draw SA′, SB′, SC′ perpendicular to the sides; then, if α, β be the semi-axes of the conic, we have $\left(\frac{\beta^2}{\delta_1}, \frac{\beta^2}{\delta_2}, \frac{\beta^2}{\delta_3}\right)$ as the distances of the *other* focus from the sides;



hence $\delta_1 \sin A + \ldots + \ldots = \frac{\beta^2}{\delta_1} \sin A + \ldots + \ldots$; $\therefore\ \beta^2 = \frac{\delta_1\delta_2\delta_3(\delta_1 \sin A + \ldots)}{\delta_2\delta_3 \sin A + \ldots}$.

Now, by the auxiliary circle property of conics, we have

α = semi-major axis = radius of circle through A′B′C′

$$= \frac{B'C' \cdot C'A' \cdot A'B'}{4 \text{ area } A'B'C'} = \frac{d_1 d_2 d_3 \sin A \sin B \sin C}{2(\delta_2\delta_3 \sin A + \ldots)};$$

therefore latus rectum $= \frac{2\beta^2}{\alpha} = \frac{4\,\delta_1\delta_2\delta_3(\delta_1 \sin A + \ldots)}{d_1 d_2 d_3 \sin A \sin B \sin C}$;

but $\delta_1 \sin A + \ldots = \frac{\delta_1 a + \ldots}{2R} = \frac{2\Delta}{2R} = 2R \sin A \sin B \sin C$;

therefore $l = 8R \frac{\delta_1\delta_2\delta_3}{d_1 d_2 d_3}$.

6238. By R. KNOWLES, B.A., L.C.P.)—A paraboloid, of parameter m, when prevented from sliding down a plane whose inclination is α, just stands on its base without falling over; prove that the length of its axis is $9m \cot^3 \alpha$.

Solution by A. ANDERSON, M.A.; C. BICKERDIKE, L.C.P.; *and others.*

Let h equal height of axis, then, if l equal radius of base, we have

$$l^2 = mh, \quad \tan\alpha = \frac{3l}{h}; \text{ whence } h = 9m \cot^2 \alpha.$$

6266. (By E. W. SYMONS, M.A.)—Prove that the tetrahedron of reference, and the tetrahedron formed by the tangent planes at its vertices to the conicoid $(l+m+n)(lyz+mzx+nxy) = mn\,xw + nl\,yw + lm\,zw$, are such that the lines joining corresponding vertices meet in a point; and that when l, m, n vary, this point moves on the surface

$$x^{-1} + y^{-1} + z^{-1} + w^{-1} = 0.$$

Solution by J. E. A. STEGGALL, B.A.; J. W. RUSSELL, M.A.; *and others.*

Using $l+m+n=-r$, the conicoid is

$$\frac{yz}{mn}+\ldots+\ldots+\frac{xw}{lr}+\ldots,$$

or $\left(\text{if } \frac{x}{e}=\text{X, etc.}\right)$ $\text{YZ}+\text{ZX}+\text{XY}+\text{XW}+\text{WY}+\text{WZ}=0$;

and the tangent planes at A, B, C, D are

$\text{Y}+\text{Z}+\text{W}=0$, $\text{X}+\text{Z}+\text{W}=0$, $\text{X}+\text{Y}+\text{W}=0$, $\text{X}+\text{Y}+\text{Z}=0$...(1, 2, 3, 4);

and the line through A and the intersection of the planes (2), (3), (4) is

$$\text{Y}=\text{Z}=\text{W}, \text{ that is } \frac{y}{m}=\frac{z}{n}=\frac{w}{r};$$

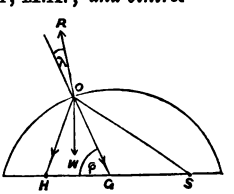
and similarly for the others; and they clearly meet in the point

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=\frac{w}{r}; \text{ therefore } x+y+z+w=0, \text{ if } l, m, n \text{ vary.}$$

6148. (By D. EDWARDES.)—A ring of weight W is just capable of sliding upon a rough semi-elliptic wire, and has attached to it two strings, which, passing through smooth small rings on the foci, support equal weights (P). If 2θ be the angle between the strings in a limiting position of equilibrium, prove that $\text{W}^2\sin^2\theta+\mu^2(\text{W}\sin\theta+2\text{P}e\cos\theta)^2=e^2\text{W}^2$, where μ is the coefficient of friction, and e eccentricity of ellipse.

Solution by J. L. KITCHIN, M.A.; G. TURRIFF, M.A.; *and others.*

Let O be the position of the ring, and ϕ the inclination to the axis of the normal at O; then, λ being the angle of friction, we have, for the equilibrium of the ring,



$$\frac{\text{W}}{\sin\lambda}=\frac{2\text{P}\cos\theta}{\cos(\phi+\lambda)},$$

or $$\frac{\text{W}}{\mu}=\frac{2\text{P}\cos\theta}{\cos\phi-\mu\sin\phi} \quad \ldots\ldots\ldots (1).$$

The geometrical equation is

$$\text{HG}=e\,.\,\text{OH}, \quad \text{or} \quad \sin\theta=e\sin\phi \quad \ldots\ldots\ldots (2).$$

Eliminating ϕ between (1) and (2), we have the result in question.

5926. (By Professor SYLVESTER, F.R.S.)—By the potential work (in respect to an arbitrary centre) of a body describing any orbit, plane or twisted,

understand what the work would be were it moving in a circle about such centre under the influence of the forces actually soliciting it; and by the residual work understand the excess of the actual stored-up work over such potential work. Also in general let an acceleration in any quantity (q) signify its second time-derivative $\left(\frac{d^2q}{dt^2}\right)$. Prove that the residual work of a particle of unit mass at each moment of time, in respect to any centre, is equal to the acceleration in the square of the radius vector drawn to such centre. [Prof. SYLVESTER calls this the *equation of residual work*. It is substantially the principle by which Lagrange reduces to quadratures the problem of the motion of a planet about two fixed suns.]

Solution by W. J. CURRAN SHARP, M.A.

Let $\frac{d^2x}{dt^2} = X$, $\frac{d^2y}{dt^2} = Y$, $\frac{d^2z}{dt^2} = Z$ be the equations of motion; then, for motion in a circle, $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$;

therefore $x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 0$;

therefore the potential work

$$= -\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right) = -(xX + yY + zZ),$$

the actual work $= 2\int(Xdx + Ydy + Zdz)$,

and the residual work $= 2\int(Xdx + Ydy + Zdz) + (xX + yY + zZ)$,

$$= x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \tfrac{1}{2}\frac{d^2(r^2)}{dt^2}.$$

6128. (By Prof. TANNER, M.A.)—The continued fraction $\frac{a}{b-}\frac{a}{b-}\cdots\frac{a}{b-}\frac{a}{b+x}$ (r terms) being represented by u_r, find a, b so that, for all values of r, $u_r = u_{r+n}$, and verify the result by showing the form of u_r.

Solution by Prof. WOLSTENHOLME, M.A.; Prof. PRATT; *and others.*

The p^{th} convergent to the fraction $\frac{a}{b-}\frac{a}{b-}\frac{a}{b-}\ldots$, is $\frac{u_p}{v_p}$, where u_p, v_p are each of the form $\lambda\alpha^{p-1} + \mu\beta^{p-1}$, and α, β are the roots of the equation $z^2 - bz + a = 0$.

For u, we have $\lambda + \mu = a$, $\lambda\alpha + \mu\beta = ab$; therefore

$$\lambda(\beta - \alpha) = a(\beta - b) = -a\alpha, \quad \text{or} \quad \lambda = \frac{a\alpha}{\alpha - \beta}, \quad \mu = \frac{a\beta}{\beta - \alpha}.$$

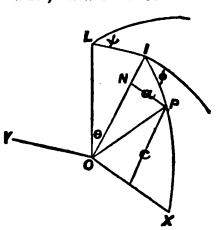
For v, we have $\lambda+\mu=b$, $\lambda a+\mu\beta=b^2-a$, and $\lambda(\beta-a)=b(\beta-b)+a=a-ba=a(\beta-b)=-a^2$, or $\lambda=\frac{a^2}{a-\beta}$, $\mu=\frac{\beta^2}{\beta-a}$; hence

$$U_r=\frac{(b+x)\frac{a(a^{r-1}-\beta^{r-1})}{a-\beta}-a^2.\frac{a^{r-2}-\beta^{r-2}}{a-\beta}}{(b+x)\frac{a^r-\beta^r}{a-\beta}-a.\frac{a^{r-1}-\beta^{r-1}}{a-\beta}}=a.\frac{a^r-\beta^r+x(a^{r-1}-\beta^{r-1})}{a^{r+1}-\beta^{r+1}+x(a^r-\beta^r)};$$

and U_r will be equal to U_{n+r}, if $a^n=\beta^n$, or $\frac{\beta}{a}=\cos\frac{2p\pi}{n}+i\sin\frac{2p\pi}{n}$, $\frac{\beta}{a}+\frac{a}{\beta}=2\cos\frac{2p\pi}{n}=\frac{b^2}{a}-2$, or $b^2=4a\cos^2\frac{p\pi}{n}$, where p is any whole number not a multiple of n. If we write $\frac{p\pi}{n}=\theta$, we may obtain U_r in the form $a^{\frac{1}{2}}\frac{a^{\frac{1}{2}}\sin r\theta+x\sin(r-1)\theta}{a^{\frac{1}{2}}\sin(r+1)\theta+x\sin r\theta}$, and writing $n+r$ for r amounts to adding $p\pi$ to each of the arguments, and therefore leaves the value unchanged.

6341. (By Prof. Minchin, M.A.)—A rigid body moves about a fixed point O, the motion of the instantaneous axis OI being completely given, as also the angular velocity of the body at each instant about it; find the components of the acceleration of any particle P, in the body along OI, a perpendicular to OI and OP, and a line perpendicular to these two directions.

Solution by G. S. Carr, B.A.; C. Morgan, B.A.; *and others.*



Let θ, ψ be polar coordinates of OI referred to a fixed line in space OL and a fixed plane through it; ϕ the angle between the planes POI and IOL; $a=$PN the perpendicular on OI; $c=$ON. Let ω_1, ω_2, ω_3 be the component angular velocities of the body about rectangular axes OX, OY, OI; the plane IOX being that of IOP. The question requires the accelerations of P in three fixed directions parallel at the moment to these moving axes.

Since OI is the instantaneous axis, $\omega_1=\omega_2=0$, and ω_3 is given. Also (indicating differentiations with respect to the time by dashes) $\theta'=\psi'=0$, and $\phi'=\omega_3$. The accelerations of P parallel to OX, OY, OI will be $\quad c\omega_2'-a\omega_3^2,\quad a\omega_3'-c\omega_1',\quad -a\omega_2'$.

Taking the usual transformations of ω_1, ω_2, ω_3 below (Routh's *Rigid Dyn.*, 3rd ed., p. 191), differentiating, and putting $\theta'=\psi'=0$, we have

$$\left.\begin{aligned}\omega_1&=\theta'\sin\phi-\psi'\sin\theta\cos\phi\\ \omega_2&=\theta'\cos\phi+\psi'\sin\theta\sin\phi\\ \omega_3&=\psi'\cos\theta+\phi'\end{aligned}\right\};\quad\therefore\quad\left\{\begin{aligned}\omega_1'&=\theta''\sin\phi-\psi''\sin\theta\cos\phi,\\ \omega_2'&=\theta''\cos\phi+\psi''\sin\theta\sin\phi,\\ \omega_3'&=\psi''\cos\theta+\phi''.\end{aligned}\right.$$

These values of ω_1', ω_2', ω_3', substituted in the above expressions, give the required accelerations in terms of known quantities.

6249. (By Prof. MATZ, M.A.)—Prove that the mean minimum eccentricity of an ellipse capable of resting in equilibrium on a *rough* inclined plane is $e_1 = 2(\sqrt{2}-1)$.

Solution by G. EASTWOOD, M.A.; D. EDWARDES; *and others.*

If (a, b) be the semi-axes of an ellipse, the maximum value of the angle between the normal at any point and the central radius vector is $\cos^{-1}\frac{2ab}{a^2+b^2}$; hence, for equilibrium, if i be the inclination of the plane, we must have $\cos i \nless \frac{2ab}{a^2+b^2}$, whence the minimum value of the excentricity is $\left(\frac{2\sin i}{1+\sin i}\right)^{\frac{1}{2}}$ (MINCHIN'S *Statics*, p. 191, Ex. 15); and the mean value is

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{2\sin i}{1+\sin i}\right)^{\frac{1}{2}}di = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{2\cos i}{1+\cos i}\right)^{\frac{1}{2}}di = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1-\tan^2\tfrac{1}{2}i)^{\frac{1}{2}}di$$

$$= (\text{putting } \tan\tfrac{1}{2}i = \sin\phi)\ \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1-\tan^2\tfrac{1}{2}i)^{\frac{1}{2}}di = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{1+\sin^2\phi}$$

$$= 2(\sqrt{2}-1).$$

6407. (By Prof. CROFTON, F.R.S.)—If, in a triangle, $C = 60°$, prove that

$$\frac{1}{a+c}+\frac{1}{b+c} = \frac{3}{a+b+c}.$$

Solution by H. MURPHY; H. L. ORCHARD, M.A.; *and others.*

$$\cos C = \tfrac{1}{2} = \frac{b^2+a^2-c^2}{2ab};\quad ab = b^2+a^2-c^2;$$

$$3ab = (a+b)^2-c^2 = (a+b+c)(a+b-c);$$

therefore

$$\frac{3}{a+b+c} = \frac{a+b-c}{ab} = \frac{1}{a+c}+\frac{1}{b+c}.$$

6416. (By Dr. HOPKINSON, F.R.S.)—An endless slightly extensible strap is stretched over two equal pulleys; prove that the maximum couple which the strap can exert on either pulley is

$$\frac{2a(c+\pi a)}{c\coth\frac{1}{2}\mu\pi+\frac{2a}{\mu}}\,T,$$

where a is the radius of either pulley, c the distance of their centres, μ the coefficient of friction, and T the tension with which the strap is put on.

For v, we have $\lambda - u = h$, $\lambda x - u\beta = h^2 - z$, and

$\lambda\beta - \alpha\beta = h\beta - h^2 + z = \alpha - h\alpha = \alpha\beta - h^2 = -\alpha^2$, or $\lambda = \dfrac{\alpha^2}{\alpha-\beta}$, $\mu = \dfrac{\beta^2}{\beta-\alpha}$;

hence

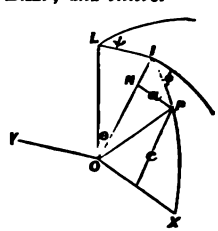
$$U_r = \frac{(h+z)\dfrac{\alpha(\alpha^{r-1}-\beta^{r-1})}{\alpha-\beta} - \alpha^2\cdot\dfrac{\alpha^{r-1}-\beta^{r-1}}{\alpha-\beta}}{(h+z)\dfrac{\alpha^r-\beta^r}{\alpha-\beta} - \alpha\beta\cdot\dfrac{\alpha^{r-1}-\beta^{r-1}}{\alpha-\beta}} = \alpha\cdot\frac{\alpha^r-\beta^r+z(\alpha^{r-1}-\beta^{r-1})}{\alpha^{r-1}-\beta^{r-1}+z(\alpha^r-\beta^r)};$$

and U_r will be equal to U_{n+r}, if $\alpha^n = \beta^n$, or $\dfrac{\beta}{\alpha} = \cos\dfrac{2p\pi}{n} + i\sin\dfrac{2p\pi}{n}$,

$\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta} = 2\cos\dfrac{2p\pi}{n} = \dfrac{h^2}{\alpha} - 2$, or $h^2 = 4\alpha\cos^2\dfrac{p\pi}{n}$, where p is any whole number not a multiple of n. If we write $\dfrac{p\pi}{n} = \theta$, we may obtain U_r in the form $\alpha^r\dfrac{\sin r\theta - x\sin(r-1)\theta}{\alpha^2\sin r\theta - x\sin(r-1)\theta}$, and writing $n+r$ for r amounts to adding $p\pi$ to each of the arguments, and therefore leaves the value unchanged.

6341. (By Prof. Mitchie, M.A.)—A rigid body moves about a fixed point O, the motion of the instantaneous axis OI being completely given, as also the angular velocity of the body at each instant about it; find the components of the acceleration of any particle P, in the body along OI, a perpendicular to OI and OP, and a line perpendicular to these two directions.

Solution by G. S. Carr, B.A.; C. Morgan, B.A.; *and others.*



Let θ, ψ be polar coordinates of OI referred to a fixed line in space OL and a fixed plane through it; ϕ the angle between the planes POI and IOL; $a = PN$ the perpendicular on OI: $c = ON$. Let $\omega_1, \omega_2, \omega_3$ be the component angular velocities of the body about rectangular axes OX, OY, OI; the plane IOX being that of IOP. The question requires the accelerations of P in three fixed directions parallel at the moment to these moving axes.

Since OI is the instantaneous axis, $\omega_1 = \omega_2 = 0$, and ω_3 is given. Also (indicating differentiations with respect to the time by dashes) $\theta' = \psi' = 0$, and $\phi' = \omega_3$. The accelerations of P parallel to OX, OY, OI will be $\quad c\omega_2' - a\omega_3^2, \quad a\omega_3' - c\omega_1', \quad -a\omega_2'$.

Taking the usual transformations of $\omega_1, \omega_2, \omega_3$ below (Routh's *Rigid Dyn.*, 3rd ed., p. 191), differentiating, and putting $\theta' = \psi' = 0$, we have

$$\left.\begin{aligned}\omega_1 &= \theta'\sin\phi - \psi'\sin\theta\cos\phi\\ \omega_2 &= \theta'\cos\phi + \psi'\sin\theta\sin\phi\\ \omega_3 &= \psi'\cos\theta + \phi'\end{aligned}\right\};\quad \therefore \left\{\begin{aligned}\omega_1' &= \theta''\sin\phi - \psi''\sin\theta\cos\phi,\\ \omega_2' &= \theta''\cos\phi + \psi''\sin\theta\sin\phi,\\ \omega_3' &= \psi''\cos\theta + \phi''.\end{aligned}\right.$$

These values of $\omega_1', \omega_2', \omega_3'$, substituted in the above expressions, give the required accelerations in terms of known quantities.

6235. (By H. McColl, B.A.)—When, from the three implications, (1) $a'b + ab' : dx$, (2) $ax + by : c$, (3) $cd : y$, may we conclude that either x or y is true, but not both?

Solution by the Proposer.

We are required to find the *weakest antecedent* of $x'y + xy'$; but it will be more convenient to find first the *strongest consequent* of $(x'y + xy')'$, that is, of $xy + x'y'$, and then transpose. Thus, using rule 23 and formula 6, we get $xy : (ab + a'b' + d)(a'b' + c)$, $x'y' : (ab + a'b')(c' + d')$.

Hence $xy + x'y' : ab + a'b' + cd$, omitting redundant terms.

Transposing (*i.e.*, by contra-position), we get

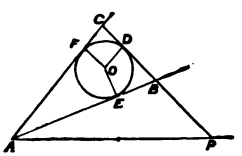
$$a'bc' + a'bd' + ab'c' + ab'd' : x'y + xy'.$$

Using rule 23 and formula 6 for the discovery of zero terms in the antecedent of this implication, we get $a'bc' : dxy'$, $a'bd' : 0$, $ab'c' : 0$, $ab'd' : 0$; hence, finally, $a'bc' : x'y + xy'$.

5086. (By the Editor.)—Through a given point P draw a straight line cutting in B and C the sides of a given angle A, in such wise that AB and AC may be together equal to BC and a given line.

Solution by J. O'Regan; E. Rutter; *and others.*

On AB and AC take AE and AF each equal to $\frac{1}{2}$M (M being the given line); at E and F draw perpendiculars to AB, AC meeting in O; then a circle drawn with O as centre and E as radius will touch AB and AC at E and F; from P draw PBC touching this circle at D, then PBC will be the line required.



For $AF + CF + AE + BE = AF + AE + BC$,

or $AB + AC = 2AE + BC = M + BC$.

5976. (By R. Tucker, M.A.)—P_1, P_2, P_3; (Q_1, Q'_1), (Q_2, Q'_2), (Q_3, Q'_3) are the respective denominators and numerators of the ultimate (proper fraction) convergents, which are obtained from making the roots of the equation $x^3 + qx + r = 0$ the quotients of a series of continued fractions; shew that

$$\frac{Q_1Q'_1}{P_1} + \frac{Q_2Q'_2}{P_2} + \frac{Q_3Q'_3}{P_3} = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3}.$$

Solution by the PROPOSER; J. O'REGAN; *and others.*

Let a, b, c be the roots of the equation; then we have

$$\frac{Q_1}{P_1} = \frac{1}{b+a+\frac{1}{c}} = \frac{ac+1}{abc+b+c}, \quad \frac{Q'_1}{P_1} = \frac{ab+1}{abc+b+c};$$

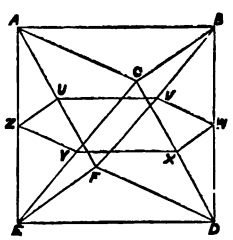
therefore $$\frac{Q_1 Q'_1 - 1}{P_1} = a, \text{ \&c.};$$

hence $$\frac{Q_1 Q'_1 - 1}{P_1} + \frac{Q_2 Q'_2 - 1}{P_2} + \frac{Q_3 Q'_3 - 1}{P_3} = a + b + c = 0.$$

6420. (By DONALD MCALISTER, D.Sc.)—A regular octohedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant.

Solution by Professor TOWNSEND, F.R.S.

Let ABCDEF (see fig.) be the six vertices of the solid, and UVWXYZ those of its hexagonal section by any plane parallel to its pair of opposite faces ABC and DEF; then, on account of the parallelism of the three tetrads of lines (AB, DE, UV, XY), (BC, EF, WX, ZU), (CA, FD, YZ, VW), and on account of the equilateralism of the eight triangular faces of the solid, we have at once, UV = VF and VW = VB, WX = XD and XY = XC, YZ = ZE and ZU = ZA; therefore, at once, perimeter of section = BF + CD + EA, = perimeter of any face of solid; and therefore, &c.



The plane of section being supposed to vary, and to assume consecutively all positions intermediate to ABC and DEF; then, on account of the three squares ABCD, BCEF, CAFD, whose planes intersect at right angles at the centre, and pass through the twelve edges of the solid, the three diagonals UX, VY, WZ of the varying section continue constantly parallel and equal to the three pairs of corresponding sides CA and FD, BC and EF, AB and DE of the two equilateral triangles ABC and DEF; and the equilateral triangle of varying magnitude they determine in every position is the intersection of their varying plane with the aforesaid triad of fixed planes intersecting at the centre and passing through the twelve edges of the solid.

6106. (By A. MARTIN, M.A.)—Show that the mean value of all the radius-vectors that can be drawn from one end of the major axis of a given ellipse to its circumference is $\frac{4b}{\pi e} \sin^{-1} e$.

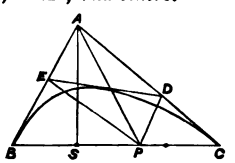
Solution by D. EDWARDES; J. A. KEALY, M.A.; *and others.*

The polar equation being $r = \dfrac{2ab^2\cos\theta}{b^2+a^2e^2\sin^2\theta}$, the required average is

$$\frac{\int r\,d\theta}{\int d\theta} = \frac{2ab^2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{d\sin\theta}{b^2+a^2e^2\sin^2\theta} = \frac{4b}{\pi e}\sin^{-1}e.$$

6402. (By W. E. WRIGHT, B.A.) — Find the envelop of the line joining the feet of the perpendiculars drawn on the sides of a right-angled triangle from any point in the hypotenuse.

Solution by C. MORGAN, B.A.; G. TURRIFF, M.A.; *and others.*



Let P be any point in BC; PD, PE perpendiculars on AC, AB; and AS a perpendicular to BC; then the circle on PA passes through E, D, S; and, similarly, if other points P′, &c. be taken, the circle on P′A will pass through E′, D′, S, &c.

Now S is a fixed point; hence DE will always touch a parabola whose focus is S; AB, AC being a pair of tangents; since in a parabola the circle through the intersection of three tangents passes also through the focus.

6034. (By Professor SYLVESTER, F.R.S.)—

$\left.\begin{array}{llll} U_1, & U_2, & \ldots & U_i \\ V_1, & V_2, & \ldots & V_j \\ W_1, & W_2, & \ldots & W_k \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right\}$ are homogeneous numerical linear functions of n variables; and $i+j+k+\ldots = n$. Any i homogeneous linear functions, in which the coefficients are all integer, are taken of the U's, j of the V's, k of the W's, and so on. Let R be the resultant of these n new linear functions of the n variables. Investigate a rule for determining the greatest common measure of the infinite number of values of R that can be thus obtained. [Prof. SYLVESTER remarks that this greatest common measure is entitled to be called the Resultant of the given functions taken *in sets.*]

Solution by W. J. C. SHARP, M.A.

Since the various new functions must all vanish, if the given functions do so, R must contain the eliminant of these as a factor; and as this is the only condition which will make all the new functions vanish, it is the only factor which is common to all the values of R, and is therefore the G.C.M. required.

6313. (By Prof. GENESE, M.A.) — Prove that, with the usual notation, the distance from the centre of the inscribed circle of a triangle to the orthocentre is $(2r^2-4R^2\cos A\cos B\cos C)^{\frac{1}{2}}$.

Solution by R. KNOWLES, B.A., L.C.P.; E. RUTTER; *and others.*

Employing trilinear coordinates, we have, for the centre of the inscribed circle, $\alpha_1=\beta_1=\gamma_1=r=\Delta s^{-1}$; and for the orthocentre,

$$\alpha_2=\frac{2\Delta\,.\cos B\,.\cos C}{c\,.\sin A\,.\sin B},\quad \beta_2=\frac{2\Delta\,.\cos A\,.\cos C}{c\,.\sin A\,.\sin B},\quad \gamma_2=\frac{2\Delta\,.\cos A\,.\cos B}{c\,.\sin A\,.\sin B};$$

$$\therefore\ d^2=\frac{abc}{4\Delta^2}\left\{a\cos A(\alpha_1-\alpha_2)^2+b\cos B(\beta_1-\beta_2)^2+c\cos C(\gamma_1-\gamma_2)^2\right\}$$

$$=\frac{abc}{4s^2}(a\cos A+b\cos B+c\cos C)-\frac{ab\cos A\cos B\cos C}{\sin A\sin B}$$

$$=\frac{abcr^2}{4\Delta^2}\cdot\frac{2\Delta}{R}-\frac{a}{\sin A}\frac{b}{\sin B}\cos A\cos B\cos C$$

$$=\frac{abcr^2}{bc\sin A}\times\frac{2\sin A}{a}-4R^2\cos A\cos B\cos C$$

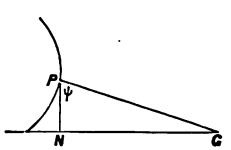
$$=2r^2-4R^2\cos A\cos B\cos C.$$

5962. (Py Professor COCHEZ.)—Trouver la courbe dont le rapport du rayon de courbure à la normale est constant et égal à m.

Solution by J. HAMMOND, M.A.; G. HEPPEL, M.A.; *and others.*

We have $\rho=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2}$,

$$PG=y\sec\psi=y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$$



Hence, if $\rho=m\text{PG}$, we have

$$\frac{d^2y}{dx^2}\div\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{1}{my}.$$

Multiplying by $2\frac{dy}{dx}$, and integrating, $\log\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{2}{m}\log y+\text{const.}$;

or, if c be a max. or min. value of y, $1+\left(\frac{dy}{dx}\right)^2=\left(\frac{y}{c}\right)^{\frac{2}{m}}$ .....(1).

Hence $x=\int dy\div\left[\left(\frac{y}{c}\right)^{\frac{2}{m}}-1\right]^{\frac{1}{2}}$ is the equation to the curve.

This equation can only be expressed in finite terms for special values of m. Equation (1) can be put in the form $y=c\sec^m\psi$.....(2).

When $m = 1$ the curve is the common catenary, and when $m = -1$ the circle, when $m = \pm \frac{1}{3}$ the integral equation contains an expression of the 4th degree under the radical, and when $m = \pm \frac{2}{3}$ a cubic expression; in these cases the relation between x and y is capable of being expressed by means of the Elliptic Functions.

The case $m = 2$ gives a parabola, whose directrix is the axis of x.

6279. (By Prof. TOWNSEND, F.R.S.)—An annular lamina of uniform density, bounded internally and externally by concentric circles of any radii, being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere on the concentric sphere, whose radius is the geometric mean between those of its bounding circles; shew, by any method, that the entire resultant attraction is perpendicular to its plane.

I. *Solution by* A. L. SELBY, M.A.; J. W. SHARPE, M.A.; *and others.*

If $(h, 0, l)$ be the attracted point, the potential is

$$mf\int_b^a\int_0^\pi \frac{r\,dr\,d\theta}{h^2+l^2+r^2-2hr\cos\theta} = mf\pi\int_b^a \frac{r\,dr}{\left\{(h^2+l^2)^2+2(l^2-h^2)r^2+r^4\right\}^{\frac{1}{2}}}$$

$$= mf\frac{\pi}{2}\log\frac{l^2-h^2+a^2+\left\{(h^2+l^2)^2+2(l^2-h^2)a^2+a^4\right\}^{\frac{1}{2}}}{l^2-h^2+b^2+\left\{(h^2+l^2)^2+2(l^2-h^2)b^2+b^4\right\}^{\frac{1}{2}}};$$

$$\therefore\ \mathrm{X} = \frac{d}{ah}(\text{potential}) = mf\pi\left\{\frac{-h+\dfrac{h(h^2+l^2)-ha^2}{\left\{(h^2+l^2)^2+2(l^2-h^2)a^2+a^4\right\}^{\frac{1}{2}}}}{l^2-h^2+a^2+\left\{(h^2+l^2)^2+2(l^2-h^2)a^2+a^4\right\}^{\frac{1}{2}}}\right.$$

$$\left.-\frac{-h+\dfrac{h(h^2+l^2)-h^2}{\left\{(h^2+l^2)^2+2(l^2-h^2)b^2+b^4\right\}^{\frac{1}{2}}}}{l^2-h^2+a^2+\left\{(h^2+l^2)^2+2(l^2-h^2)b^2+b^4\right\}^{\frac{1}{2}}}\right\},$$

which vanishes when $h^2+l^2 = ab$, *i.e.*, when the attracted point is on the concentric sphere, &c.

II. *Solution by the* PROPOSER.

Denoting by ρ the mass per unit of area of the lamina, by a and b the radii of its two bounding circles, by r and s those of any two others in its interior concentric with its mass, for which $rs = ab$, and consequently inverse to each other with respect to the sphere, by $r \pm dr$ and $s \mp ds$ those of a similarly related consecutive pair in either direction, by r_1 and r_2 and by s_1 and s_2 the greatest and least distances of any point P on the sphere from the circumferences of the two circles r and s, by x the perpendicular distance of P from the axis of the lamina, by μ the mass of the attracted particle supposed situated at P, and by f the absolute intensity of the attraction. Then, since (*Reprint*, Vol. XXXII., page 63), the attractions, parallel to the plane of the lamina, of the two slender circular rings

of its mass intercepted between the two pairs of consecutive circles, r and $r \pm dr$, s and $s \mp ds$, are equal respectively to

$$2\pi\mu f\rho \cdot \frac{r\,dr}{r_1^2 r_2^2} \cdot x \quad \text{and} \quad 2\pi\mu f\rho \cdot \frac{s\,ds}{s_1^2 s_2^2} \cdot x,$$

and act in opposite directions, the two spherical segments connecting P with the circumferences of r and s being necessarily one greater and the other less than a hemisphere; and since, by hypothesis, $dr : ds = r : s$, to prove the property for the joint attraction of the two elementary rings, and therefore for that of the two entire segments of the lamina interior and exterior to the sphere, it is consequently only necessary to show that $r_1 r_2 : s_1 s_2 = r : s$, which, as r and s are inverse circles with respect to the sphere, follows at once from a well-known elementary property of any pair of inverse points with respect to a circle.

That the attraction of the lamina, for the same law of force, is evanescent for every element of its mass situated on its circle of intersection with the sphere (*Reprint*, Vol. XXXII., page 95), is, of course, an immediate consequence from the above.

6353. (See p. 51). II. *Solution by* Rev. R. Harley, F.R.S.

In Boole's notation the four given conditions may be written thus:—

$$x = x(yz + y'z'), \quad wx = wx(yz + y'z'), \quad w'xy = w'xyx, \quad w'xy'z = 0;$$

or, $x(yz + y'z') = 0$, $wx(yz' + y'z) = 0$, $w'xyz' = 0$, $w'xy'z = 0$;

whence, adding the equations together, and remembering that $w + w' = 1$, and $yz + y'z + yz' + y'z' = 1$, we have $x = 0$.

6204. (By J. J. Walker, M.A.)—Referring to Quest. 5680, show that the conditions for the double contact of two conics—not expressible in terms of the invariants of the system—may be obtained in terms of coefficients of the three contravariant conics of the system.

I. *Solution by* Professor Wolstenholme, M.A.

The equations of S, S′, two conics having double contact, may be written (in an infinite number of ways) in the forms

$$lx^2 + my^2 + nz^2 = 0, \quad lx^2 + my^2 + n'z^2 = 0.$$

Hence the discriminant of $k\mathrm{S} + \mathrm{S}'$ is $(k+1)^2(kn + n')lm$, which has two equal roots, and it is manifest that this is the only relation between the

roots for all forms of the equations. But the discriminant will have two equal roots when the conics have single contact. Hence the conditions for double contact cannot be expressed in terms of Δ, Θ, Θ', Δ'. The conditions may be found from the fact that $kS+S'$ is a perfect square for that value of k supplied by the two equal roots. Hence, taking the general equations $(abcfgh\Box x, y, z)^2 = 0$, $(a'b'c'f'g'h'\Box x, y, z)^2 = 0$, we have

$$(ka+a')(kf+f') = (kg+g')(kh+h'),$$

$$(kb+b')(kg+g') = (kh+h')(kf+f'), \quad (kc+c')(kh+h') = (kf+f')(kg+g');$$

which, eliminating k, will give two conditions which must include the one relation between the invariants

$$(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta)+4(\Theta\Theta'-9\Delta\Delta')^2 = 0.$$

I must own I do not see the *à priori* reason why the two conditions cannot be expressed in terms of the invariants, and I asked the question in the Senate-House Examinations for 1870; but this was the one question left unanswered. It is obvious that the equations given in Vol. XXXI, p. 84, are not *homogeneous* in the proper sense.

Any equation between Δ, Θ, Θ', Δ', expressing a projective relation between the two conics, will be unaltered if we write $\lambda^3\Delta$, $\lambda^2\mu\Theta$, $\lambda\mu^2\Theta'$, $\mu^3\Delta'$ for Δ, Θ, Θ', Δ'. I have never succeeded in obtaining the true conditions in a form which I thought worthy of publication.

[The solution of Quest. 5680, given on p. 86 of Vol. XXXI. of our *Reprints*, is erroneous, in assuming $S-S'$ to be a line-pair.]

II. *Solution by the* PROPOSER.

The equations in k of Prof. WOLSTENHOLME's Solution are, in effect,

$$Fk^2+F_1k+F' = 0, \quad Gk^2+G_1k+G' = 0, \quad Hk^2+H_1k+H' = 0,$$

where $F = gh-a'f$, $F' = g'h'-a'f'$, $F_1 = gh'+g'h-af'-a'f$.....;

and the conditions that these equations shall hold simultaneously for a common value of k, form a solution of the Question proposed, *e.g.*, the condition that the first two should have a common root, combined with the condition that it should satisfy the third. I quite assent to Prof. WOLSTENHOLME's remark as to the absence of any *à priori* reason, &c.

6378. (By Prof. WOLSTENHOLME, M.A.)—Prove that, if $n > 1$,

$$\int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}} = \tfrac{1}{4}\pi \left\{ \frac{\Gamma\frac{1}{2}(n-1)}{\Gamma\frac{1}{2}(n)} \right\}^2.$$

Solution by H. STABENOW, M.A.; A. L. SELBY, B.A.; *and others.*

From the identity

$$(1-x)^{-\kappa} = 1+\kappa x+\frac{\kappa(\kappa+1)}{1.2}x^2+\ldots+\frac{\Gamma(\kappa+m)}{\Gamma(\kappa)\,\Gamma(m+1)}x^m+\ldots = \Sigma A_{\kappa,m}x^m$$

we have

$$U = \int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}}$$

$$= \Sigma \int_0^1 A_{\frac{1}{2}n,m}(1-x)^{n-2}x^{2m}\,dx \int_0^{\frac{1}{2}\pi} \sin^{2m}\theta\,d\theta$$

$$= \Sigma \int_0^1 A_{\frac{1}{2}n,m} \left\{ \frac{1}{2}\,\frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2})}{\Gamma(m+1)} \right\} (1-x)^{n-2}x^{2m}\,dx$$

$$= \tfrac{1}{4}\pi\,\Sigma \left\{ A_{\frac{1}{2}n,m}\,A_{\frac{1}{2},m}\,\frac{\Gamma(n-1)\,\Gamma(2m+1)}{\Gamma(n+2m)} \right\}.$$

Now $A_{\frac{1}{2}n,m}\,\dfrac{\Gamma(n-1)\,\Gamma(2m+1)}{\Gamma(n+2m)} = \dfrac{\frac{1}{2}n.\frac{1}{2}n+1.\frac{1}{2}n+2\ldots\frac{1}{2}n+m-1}{1.2.3\ldots m}$

$$\times \frac{1.2.3\ldots 2m}{n-1.n.n+1\ldots n+2m-1} = \frac{1.3.5\ldots 2m-1.n.n+2\ldots n+2m-2}{n-1.n.n+1.n+2\ldots n+2m-1}$$

$$= \frac{1.3.5\ldots 2m-1}{n-1.n+1.n+3\ldots n+2m-1} = \frac{1}{2}\,\frac{\Gamma[\frac{1}{2}(n-1)]\,\Gamma(m+\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)+m]\,\Gamma(\frac{1}{2})};$$

therefore $U = \tfrac{1}{4}\pi\,\Sigma \left\{ A_{\frac{1}{2},m}\,\dfrac{\Gamma[\frac{1}{2}(n-1)]\,\Gamma(m+\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)+m]\,\Gamma(\frac{1}{2})} \right\}$

$$= \tfrac{1}{4}\pi\,\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)}\,\Sigma \int_0^1 \frac{(1-x)^{\frac{1}{2}n-\frac{1}{2}}x^{m-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,A_{\frac{1}{2},m}\,dx$$

$$= \tfrac{1}{4}\pi\,\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)} \int_0^1 \frac{(1-x)^{\frac{1}{2}(n-1)-1}x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,dx = \tfrac{1}{4}\pi \left\{ \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)} \right\}^2.$$

6293. (By J. W. Russell, M.A.)—Two bags A and B contain each two balls, each of which is either white or black; a ball is drawn from A and put into B, and then a ball is drawn from B and put into A; in both cases the ball turns out to be white; and this double operation is repeated n times; find the probability of its not failing in the $(n+1)^{\text{th}}$ attempt.

Solution by C. J. Monro, M.A.; J. E. W. Steggall, M.A.; *and others.*

It will, I think, be according to the Proposer's intention, if I lay especial stress on the relation of this problem to first principles, by estimating how much assumption is made at each point. Each double operation, resulting as given, leaves the distribution unaltered. Let U_x, V_x, W_x, for A, and u_x, v_x, w_x, for B, be the chances of two white balls, one, none, after the x^{th} operation. Then the answer,—call it P_n,—is

$$(U_n + \tfrac{1}{2}V_n + 0W_n)(u_n + \tfrac{2}{3}v_n + \tfrac{1}{3}w_n),$$

or

$$\frac{1}{2.3}(2U_n + V_n + 0W_n)(3u_n + 2v_n + w_n).$$

In general, also, if $S^{-1}(a, b, \ldots)$ stand for $a, b, \ldots$ divided by their sum, then

$$U_x, V_x, W_x = S^{-1}(2U_{x-1}, V_{x-1}, 0W_{x-1}),$$

and

$$u_x, v_x, w_x = S^{-1}(3u_{x-1}, 2v_{x-1}, w_{x-1}).$$

But this may conceivably not hold when $x = 1$. For the first stage of the

first operation (the only one, as given, which has an unprecedented result) tells us that there is at least one white ball, and this might affect our judgment on the colours of the rest. U_1, &c. would then be arbitrary; but in any case U_x, &c. are $S^{-1}(2^{x-1}U_1, V_1, 0^{x-1}W_1)$, $S^{-1}(3^{x-1}u_1, 2^{x-1}v_1, w_1)$.

It is natural to assume that the colour of one ball is no evidence of another's, and thus U_1, &c., given in terms of U_0, &c.; also that $W_0 = U$, $w_0 = u_0$. More arbitrary are $V_0 : U_0$, $v_0 : u_0$; two hypothetical values are obvious, 1 and 2.* We have in these cases, for P_n,

$$\frac{1}{2.3} \cdot \frac{2^{n+1}+1+0^{n+1}}{2^n+1+0^n} \cdot \frac{3^{n+1}+2^{n+1}+1}{3^n+2^n+1},$$

$$\frac{1}{2.3} \cdot \frac{2^{n+1}+2+0^{n+1}}{2^n+2+0^n} \cdot \frac{3^{n+1}+2^{n+2}+1}{3^n+2^{n+1}+1}.$$

$P_0 = \frac{1}{3}$ in each case; for, when $n = 0$, $0^n = 1$, n being the number of times W_0 is multiplied by 0. When $n > 0$, we may write for P_n

$$\frac{1}{2.3} \cdot \frac{2^{n+1}+1}{2^n+1} \cdot \frac{3^{n+1}+2^{n+1}+1}{3^n+2^n+1}, \quad \frac{1}{2.3} \cdot \frac{2^n+1}{2^{n-1}+1} \cdot \frac{3^{n+1}+2^{n+2}+1}{3^n+2^{n+1}+1}.$$

[* See *Reprint*, Vol. XXXI., pp. 100—103, and XXXIII., pp. 107—110].

II. *Solution by the* Proposer.

The contents of the bags A and B at first are

(ww) (ww) (ww) (wb) (wb) (wb) ..... (A),

(ww) (wb) (bb) (ww) (wb) (bb) ..... (B).

The chances of drawing white from A are

$$1, 1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2};$$

hence probabilities of the six hypotheses are

$$\tfrac{2}{9}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}.$$

The state of the second bag is now

$$(www)\,(wbw)\,(bbw)\,(www)\,(wbw)\,(bbw),$$

and the chances of drawing a white from B are

$$1, \tfrac{2}{3}, \tfrac{1}{3}, 1, \tfrac{2}{3}, \tfrac{1}{2}.$$

Hence, probabilities of the six hypotheses as to contents of B are

$$\tfrac{3}{12}, \tfrac{3}{12}, \tfrac{1}{12}, \tfrac{3}{12}, \tfrac{2}{12}, \tfrac{1}{12}.$$

Similarly, it is easily seen that the probabilities of the above twelve hypotheses concerning the state of the bags are, after the nth double event, in the ratios of

$$2^n : 2^n : 2^n : 1 : 1 : 1,$$

and of

$$3^n : 2^n : 1 : 3^n : 2^n : 1.$$

The probability of the happening of the event the $(n+1)^{\text{th}}$ time, is therefore the product of

$$\frac{2^n}{3(2^n+1)} \times 3 + \frac{1}{3(2^n+1)} \times \frac{3}{2}$$

by

$$\frac{3^n}{2(3^n+2^n+1)} \times 2 + \frac{2^n}{2(3^n+2^n+1)} \times \frac{4}{3} + \frac{1}{2(3^n+2^n+1)} \times \frac{2}{3};$$

i.e., the required chance is

$$\frac{2^n+\frac{1}{2}}{2^n+1} \cdot \frac{3^n+\frac{1}{3}.2^{n+1}+\frac{1}{3}}{3^n+2^n+1},$$

which is equal to 1 (as it should be) when $n = \infty$.

for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle.

Solution by G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others.*

1. We know (SALMON'S *Higher Curves*) that $(2n-1)$ tangents of a curve of the n^{th} class, among whose tangents the line at infinity counts for $(n-1)$, being given, that the locus of the focus is a circle; hence the $2n$ circles, so determined from $2n$ lines by leaving out one at a time, must meet in a point which is the focus of the curve of the same kind touching all the $2n$ lines.

2. Again, $2n+1$ tangents of a curve of the $(n+1)^{\text{th}}$ class, among whose tangents the line at infinity counts for n, being given, the locus of the focus is a circle. Hence, taking as a complex the curve of the n^{th} class touching $2n$ of the $(2n+1)$ lines and the point at infinity on the remaining one, the focus of the complex will be the focus of the curve, and the $(2n+1)$ points so determined lie on a circle.

6439. (By Prof. CROFTON, F.R.S.)—Prove that

$$\int e^{\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}x^2}(x^{-1} + 1\,.\,x^{-3} + 1\,.\,3x^{-5} + 1\,.\,3\,.\,5x^{-7} + \ldots).$$

Solution by J. HAMMOND, M.A.; G. HEPPEL, M.A.; *and others.*

If $u_n = \int e^{\frac{1}{2}x^2} x^n\,dx$, integrating by parts, we have

$$\begin{aligned} u_n &= e^{\frac{1}{2}x^2}x^{n-1} - (n-1)\int e^{\frac{1}{2}x^2}x^{n-2}\,dx \\ &= e^{\frac{1}{2}x^2}\{x^{n-1} - (n-1)\,x^{n-3} + (n-1)\,(n-3)\,x^{n-5} - \&c.\}. \end{aligned}$$

The series in the question is obtained by putting $n = 0$.

6488. (By W. B. GROVE, B.A.)—If my son is to enter either the law or the church, he must go either to Oxford or to Cambridge. If he goes to Oxford without entering the law, or to Cambridge without entering the church, he will get a legacy at his uncle's death. He will fail to get the legacy under the following circumstances:—If he will not go to Oxford, and at the same time will not enter the church; or if he will not go to Cambridge, and at the same time will not enter the law—and under *no other* circumstances. I decide that he must enter either the law or the church: will he or will he not get the legacy?

Solution by ELIZABETH BLACKWOOD.

Let α, β, m, n, x respectively denote the statements: He will enter the *law*, He will enter the *church*, He will go to *Oxford*, He will go to *Cambridge*, He will get the *legacy*. The data will then be

$$(\alpha+\beta : m+n)\ (m\alpha'+n\beta' : x)\ (x' = m'\beta'+n'\alpha')\ (\alpha+\beta)\ (\alpha : \beta')\ (m : n'),$$

the last two factors being understood, though not expressed in the enunciation of the problem. We get, by inspection,

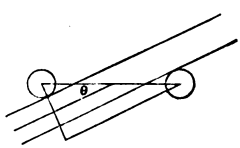
$$x' : (\alpha'\beta'+m+n)\,(m'+\alpha)\,(n'+\beta)\ (m'\beta'+n'\alpha')\ (\alpha+\beta)\,(\alpha'+\beta')\,(m'+n').$$

In the consequent of this implication, the last two factors are implied in the fourth, and may therefore be omitted. The product of the rest is zero. Hence $1 : x$, so that he will certainly get the legacy.

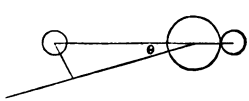
6447. (By CHRISTINE LADD.)—The width of a croquet hoop, the thickness of its wires, and the diameter of a ball are given: the ball being in a given position, show how to find the conditions that it may just be possible for it to go through the hoop, (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection.

Solution by G. F. WALKER, M.A.; H. T. GERRANS, B.A.; *and others.*

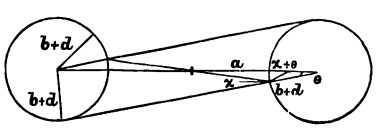
Let b be the radius of a wire, $2a$ the distance between the centres of the wires, $2d$ the diameter of the ball.



1. The line joining the centre of the ball to the centre of the distance $2a$ makes an angle $\sin^{-1}\dfrac{b+d}{a}$ with $2a$.



2. The tangent from the centre of the ball to a circle of radius $b+d$, with its centre at centre of nearer wire, $\sin^{-1}\dfrac{b+d}{2a-b-d}$ with the distance $2a$.



3. The tangent from the centre of the ball to the circle of radius $b+d$, with its centre at centre of nearer wire, makes an angle $\chi+\theta$ with the distance $2a$, χ and θ being given by the equations

$$\frac{b+d}{\sin(\chi-\theta)} = \frac{a}{\sin\chi}, \qquad \frac{b+d}{\sin(\chi+\theta)} + \frac{(b+d)\sin\chi}{\sin(\chi+\theta)} = 2a.$$

If the angle the tangent makes with $2a$ be less than this, the ball cannot get through without striking a wire more than once; and if the angle be less than $\sin^{-1}\dfrac{b+d}{2a}$, the ball cannot get through at all.

There is another limit, however; for, if $\chi+\theta$ be less than the result found from solving the equations

$$\frac{b+d}{\sin(\chi-\theta)}=\frac{2a}{\sin\chi},\quad \frac{b+d}{\sin(\chi+\theta)}+\frac{(b+d)\sin\chi}{\sin(\chi+\theta)}=2a,$$

the ball cannot get through, but will bound outwards from the nearer wire.

6479. (By Prof. WOLSTENHOLME, M.A.)—Prove that the asymptotic cone of a quadric $u=0$ is $A\left(\frac{\delta u}{\delta x}\right)^2+2F\frac{\delta u}{\delta y}\frac{\delta u}{\delta z}+\&c.=0$, and the equations of its axes are

$$\frac{A\frac{\delta u}{\delta x}+H\frac{\delta u}{\delta y}+G\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta x}}=\frac{H\frac{\delta u}{\delta x}+B\frac{\delta u}{\delta y}+F\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta y}}=\frac{G\frac{\delta u}{\delta x}+F\frac{\delta u}{\delta y}+C\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta z}},$$

where $A=\frac{\delta^2 u}{\delta y^2}\frac{\delta^2 u}{\delta z^2}-\left(\frac{\delta^2 u}{\delta y\,\delta z}\right)^2$, $F=\frac{\delta^2 u}{\delta z\,\delta x}\frac{\delta^2 u}{\delta x\,\delta y}-\frac{\delta^2 u}{\delta y\,\delta z}\frac{\delta^2 u}{\delta x^2}$, &c.

Solution by Professor TOWNSEND, F.R.S.

Since, for every point (x, y, z) at which the asymptotic cone, or an axis of figure, meets a quadric, the tangent plane in the former case, and the normal line in the latter case, passes through the centre $x_0y_0z_0$ of the surface, we have consequently, between the two triads of coordinates, xyz and $x_0y_0z_0$, in the former case the relation

$$(x-x_0)\frac{du}{dx}+(y-y_0)\frac{du}{dy}+(z-z_0)\frac{du}{dz}=0,$$

and in the latter case the relations

$$(x-x_0):(y-y_0):(z-z_0)=\frac{du}{dx}:\frac{du}{dy}:\frac{du}{dz};$$

from which, since for every point xyz on the quadric, as appears at once from the known equations for the coordinates $x_0y_0z_0$ of its centre,

$$(x-x_0)D=A\frac{du}{dx}+H\frac{du}{dy}+G\frac{du}{dz},\quad (y-y_0)D=B\frac{du}{dy}+F\frac{du}{dz}+H\frac{du}{dx},$$
$$(z-z_0)D=C\frac{du}{dz}+G\frac{du}{dx}+F\frac{du}{dy},$$

where $D=abc+2fgh-af^2-bg^2-ch^2$; therefore &c., as regards both parts of the question.

Denoting by k the common value of the three equal ratios in the above equations for the axes of the surface, since then

$$(A-K)\frac{du}{dx}+H\frac{du}{dy}+G\frac{du}{dz}=0,\quad H\frac{du}{dx}+(B-K)\frac{du}{dy}+F\frac{du}{dz}=0,$$
$$G\frac{du}{dx}+F\frac{du}{dy}+(C-K)\frac{du}{dz}=0;$$

and since, consequently, $\begin{vmatrix} (A-K) & H & G \\ H & (B-K) & F \\ G & F & (C-K) \end{vmatrix} = 0$, we have, accordingly, for the determination of the three values of k, the well-known cubic determinant for the directions of the axes of the reciprocal surface. [See SALMON'S *Geometry of Three Dimensions*, General Theory of Quadrics.]

If $u = 0$ represent originally a cone having its vertex at the origin, since then the two cones

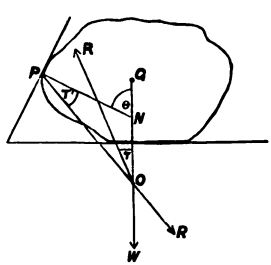
$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

are reciprocal polars to each other with respect to the point sphere having its centre at their common vertex $x^2 + y^2 + z^2 = 0$, and since the three surfaces have in consequence a common triad of principal planes, constituting in their case the Jacobian of the system, the equation of the triad for either of the two cones is accordingly $\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ x & y & z \end{vmatrix} = 0.$

[See SALMON'S *Geometry of Three Dimensions*, General Theory of Quadric Surfaces, Note.]

6442. (By Prof. MINCHIN, M.A.)—A body of any shape, with a plane base, rests with this base on a rough horizontal plane; a heavy beam movable round a horizontal axis fixed in the plane rests at a single point against the body, the vertical plane through the beam containing the centre of gravity of the body. Show that limiting equilibrium of the system is impossible unless the normal to the surface of contact of the beam and body makes with the vertical an angle greater than the sum of the angles of friction between the body and the beam and the body and the ground.

Solution by D. EDWARDES; G. HEPPEL, M.A.; *and others.*



Let the total resistance of the ground meet the vertical through the centroid of the body in O. Let P be the point of contact of the beam with the body, PN the normal to the surface of contact, θ the inclination of PN to the vertical. Then the total resistance at P must pass through O, and make with the vertical an angle greater than λ; that is, we must have $\theta - \lambda' > \lambda$, or $\theta > \lambda' + \lambda$.

[Prof. MINCHIN states that he ought to have limited the question to equilibrium which is to be broken by the slipping of the body along the ground. It will be easily seen that, if the body can topple over, so that slipping takes place only at P, the result stated in the problem is not necessary.]

6417. (By C. J. Monro, M.A.)—The birds carry away the seed as fast as it falls, but the sower has the start of them by n grains. What is the probability that, after they have carried away x grains, y of those left are of the original n? (See *Origin of Species*, 5th ed., p. 381.)

I. *Solution by the* Rev. A. Law Wathberston, M.A., F.R.A.S.

At any time, if p original grains be left, the chance of one being taken is $\frac{p}{n+1}$; $\frac{n+1-p}{n+1}$ being the chance of a fresh grain being taken.

Suppose, then, that a fresh grain is taken α times in succession; then an original one; then a fresh one β times in succession; then an original one; and so on. The chance in this instance is

$$\frac{1}{(n+1)^x} \cdot n \cdot \overline{n-1} \ldots (y+1) \cdot 1^\alpha \cdot 2^\beta \cdot 3^\gamma \ldots (n+1-y)^\lambda,$$

where $$\alpha+\beta+\gamma \ldots +\lambda = x-(n-y).$$

The total chance is therefore

$$\frac{1}{(n+1)^x} \cdot n \cdot \overline{n-1} \ldots (y+1) \, \Sigma \cdot 1^\alpha \cdot 2^\beta \cdot 3^\gamma \ldots (n+1-y)^\lambda,$$

the summation including all values of $\alpha, \beta, \gamma \ldots$ subject to the above condition.

The sum of the probabilities for all values of y being unity, we can find Σ by expanding $(n+1)^x$ in a series of factorials. Thus, putting $\phi(n)$ for $(n+1)^x$, we have

$$(n+1)^x = 1 + n \cdot \Delta\phi(0) + \frac{1}{2!} n^{(2)} \Delta^2\phi(0) + \ldots$$

The chance is therefore $\frac{1}{(n+1)^x} \cdot \frac{n!}{y!\,(n-y)!} \Delta^{n-y}\phi(0)$.

II. *Solution by* G. S. Carr, B.A.

For clearness suppose the n original grains to be white, and the rest black. Let $n-y=r$; then the probability of r black grains remaining is required.

A black grain falls; then the number of ways in which a grain can be carried off is $n+1$. Similarly, when a second grain falls. Thus, when x black grains have been sown, the number of ways in which x grains of either sort can have been carried away is $(n+1)^x$.

These are the unrestricted cases, the selection of any one grain at each juncture being the equally likely event with which the restricted events are now to be compared.

Let $\mathrm{B}(r, x)$ stand for *the number of ways in which r black grains can remain after x have been sown.* Now, if, when r black grains remain, a black grain be added, then a black grain can be carried off in $r+1$ ways; and if, when $r-1$ black grains remain, a black grain be added, then a white grain can be carried off in $n-r+1$ ways; in each case *leaving r black grains.* Hence the following relation subsists

$$\mathrm{B}(r, x) = (r+1)\,\mathrm{B}(r, x-1) + (n-r+1)\,\mathrm{B}(r-1, x-1) \ldots\ldots (1).$$

Applying this formula *ab initio*, observing that, from the nature of the

case, $B(r, x)$ is always zero when r is $> x$, and that $B(0, x)$ is always unity, we shall obtain the general value

$$B(r, x) = C_{xr}\{(r+1)^x - C_{r1}r^x + C_{r2}(r-1)^x - C_{r3}(r-2)^x + \ldots \pm 1\} \ldots (2),$$

the truth of which may be proved inductively by substituting from it the values of $B(r, x-1)$ and $B(r-1, x-1)$ in equation (1).

The probability of exactly r black grains remaining is

$$= B(r, x) + (n+1)^x.$$

The probability of r black grains or fewer, that is, of y white grains or more, remaining is

$$= \{B(0, x) + B(1, x) + B(2, x) + \ldots + B(r, x)\} + (n+1)^x,$$

with the values from formula (2) substituted in each case.

6461. (By H. McColl, B.A.)—Show that the weakest addition that must be made to the premises A : a, B : b, C : c, &c., to justify the inference Q : q, is the implication $Qq' : Aa' + Bb' + Cc' + \ldots$ [One of the problems solved by Mr. Venn, in the *Philosophical Magazine* for July, in illustration of his diagrammatic method, is a particular case of this general theorem.]

Solution by W. B. Grove, B.A.; Elizabeth Blackwood; *and others.*

To justify the inference Q : q, we must have $Qq' : 0$. Now, from the data $Aa' : 0$, $Bb' : 0$, &c., and the statement $Aa' + Bb' + Cc' + \ldots$ is weaker than any one or more of its terms, short of the whole; therefore the weakest additional datum necessary is

$$Qq' : Aa' + Bb' + Cc' + \ldots$$

6091. (By R. Tucker, M.A.)—If a point be taken on a certain line and normals be drawn from it to an ellipse, a circle can be drawn through the intersections of the normals with the ellipse. Find the envelop of this circle.

Solution by G. Eastwood, M.A.; *the* Proposer; *and others.*

The equation to the normal from (x', y') is

$$ax' \sin\phi - by' \cos\phi = c^2 \sin\phi \cos\phi;$$

that is, $$c^4z^4 - 2ac^3x'z^3 + (a^2x'^2 + b^2y'^2 - c^4)z^2 + 2ax'c^2z - a^2x'^2 = 0 \ldots\ldots\ldots(1),$$

where $$z \equiv \cos\phi.$$

Since the point (ϕ) is on the circle, we have

$$a^2\cos^2\phi + b^2\sin^2\phi + Aa\cos\phi + Bb\sin\phi + C = 0,$$

or $c^4z^4 + 2Aac^2z^3 + [A^2a^2 + B^2b^2 + 2c^2(C+b^2)]z^2$
$+ 2Aa(C+b^2)z + (C+b^2)^2 - B^2b^2 = 0$.....(2).

From identity of (1) and (2), we get

$A = -x'$, $ax'c^2 = Aa(C+b^2)$; therefore $C = -a^2$, $B^2b^2 = c^4 + a^2x'^2$.

Also, equating coefficients of z^2, we have $a^2x'^2 = b^2y'^2$; hence (x', y') must lie on one of the lines $by' = \pm ax'$.

The equation to the circle is $x^2 + y^2 - x'x + \frac{(c^4 + a^2x'^2)^{\frac{1}{2}}}{b}y - a^2 = 0$...(3).

For the envelop, we get

$x'^2 = b^2c^4x^2 \div [a^2(a^2y^2 - b^2x^2)]$, $c^4 + a^2x'^2 = a^2c^4y^2 \div (a^2y^2 - b^2x^2)$;

and (3) becomes the quartic

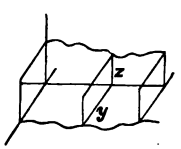
$$a^2b^2(x^2+y^2)^2 - 2a^4b^2(x^2+y^2) + a^6b^2 = c^4(a^2y^2 - b^2x^2).$$

6443. (By Prof. TAIT, M.A.)—Show that, whatever functions of x be represented by y and z, we have always $\frac{\int yz\,dx}{\int y\,dx} > e^{\frac{\int y \log z\,dx}{\int y\,dx}}$, all the integrals being taken between the same limits of x; and all the quantities involved being positive.

Solution by G. F. WALKER, M.A.; T. W. OPENSHAW, M.A.; *and others.*

Consider the volume bounded by the surfaces

$$y\,[= f(x)],\quad z\,[= \phi(x)].$$



The volume of this is $\int yz\,dx$, and the area of base is $\int y\,dx$. The given inequality is the same as

$$\log\frac{\int yz\,dx}{\int y\,dx} - \frac{\int y\log z\,dx}{\int y\,dx} > 0, \quad \log\bar{z} - \frac{\int y\log z\,dx}{\int y\,dx} > 0,$$

where $\bar{z}$ is the mean value of z for points in the area.

The second quantity is the mean value of $\log z$, which is of course less than the log of mean value of z. Hence the result.

6335. (By W. E. WRIGHT, B.A.)—If ds be an element of an arc of an ellipse whose semi-axes are a, b; and a', b' the semi-axes of the confocal hyperbola through the element; prove that $\frac{ds}{da'} = \left(1 + \frac{b^2}{b'^2}\right)^{\frac{1}{2}}$.

Solution by J. E. STEGGALL, M.A.; Prof. MATZ, M.A.; *and others.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1,$$

$$x^2=\frac{a^2(a^2-\lambda^2)}{a^2-b^2}=\frac{a^2a'^2}{a^2-b^2}, \quad y^2=\frac{b^2b'^2}{a^2-b^2};$$

therefore $(dx)^2+(dy)^2=\frac{a^2(da')^2}{a^2-b^2}+\frac{b^2(db')^2}{a^2-b^2}$, but $a'da'=-b'db'$;

therefore $$ds^2=\frac{da'^2}{a^2-b^2}\left(a^2-b^2\frac{a'^2}{b'^2}\right)=da'^2\left(1+\frac{b^2}{b'^2}\right).$$

6435. (By W. E. WRIGHT, B.A.)—Eliminate α between the equations

$$(x^2\cos\beta+\cos\alpha)\cos\alpha+(x+\sin\alpha)\sin\alpha+x\sin\beta=0,$$
$$(x^2\cos\alpha+\cos\gamma)\cos\gamma+(x+\sin\gamma)\sin\gamma+x\sin\alpha=0.$$

Solution by C. MORGAN, B.A.; J. O'REGAN; *and others.*

Subtracting the equations, we get

$$x\cos\alpha\sin\tfrac{1}{2}(\beta+\gamma)=\cos\tfrac{1}{2}(\beta+\gamma).$$

Substituting in the first, we get

$$x\cos\beta\cot\tfrac{1}{2}(\beta+\gamma)+1+x\sin\alpha+x\sin\beta=0;$$

and therefore $$\left(x\frac{\cos\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}(\beta+\gamma)}+1\right)^2+\cot^2\tfrac{1}{2}(\beta+\gamma)=x^2,$$

$$x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0.$$

6444. (By Prof. MATZ, M.A.)—Three points taken at random, one on each side of a plane triangle, are joined by straight lines; show (1) that the mean area of the triangle thus formed is one-fourth of the area of the triangle; and (2) generalize the problem for any number of points taken in different areas or volumes.

Solution by A. MARTIN, M.A.; J. O'REGAN; *and others.*

1. The mean will evidently be when each point is in its mean position, *i.e.*, at middle point of side, and therefore mean value is one-fourth of the area of the triangle.

2. Generally, if a number of points be confined to different areas or volumes which have no parts common, the mean value of the area or the solid formed by joining them, is the value of the area or solid formed by joining the centres of gravity of the areas or volumes to which the points are confined. [See *Reprint*, Vol. I., p. 50, Art. 6 of Solution of Quest. 1282.]

6066. (By Prof. WOLSTENHOLME, M.A.)—A heavy particle moves on the interior of a smooth paraboloid whose axis is vertical, and z_1, z_2 are its least and greatest vertical heights above the vertex, ρ the radius of absolute curvature of its path when at a vertical height z: prove that

$$\frac{4a(a+z)^3}{z}\,\frac{(z_1+z_2-z)^2}{\rho^2} = \frac{\{(z_1+z_2-z)[a(z_1+z_2-z)+z_1z_2]^2+az_1z_2(a+z)^2\}^2}{z(z_1+z_2-z)^2[a(z_1+z_2-z)+z_1z_2]^2+az_1^2z_2^2(a+z)^2}.$$

[As a test, suppose $z_1=0$, so that the path is plane, and we have $\frac{4(z_2-z)^2}{z}\,\frac{a(a+z)^3}{\rho^2} = \frac{[(z_2-z)^3a^2]^2}{z(z_2-z)^4a^2}$, or $\rho^2 = \frac{4(a+z)^3}{a}$, which is correct.]

Solution by the PROPOSER.

For the radii of absolute curvature at the highest and lowest points ρ_2, ρ_1, we have $\rho_1^2 = \frac{4az_2^2(a+z_1)}{z_2^2+az_1}$, $\rho_2^2 = \frac{4az_1^2(a+z_2)}{z_1^2+az_2}$;

and if $z_1=z_2$, each of these becomes $4az_1$, the particle then describing a circle of radius $2(az_1)^{\frac{1}{2}}$.

The angle ϕ described about the axis by the projection of the particle on a horizontal plane is given by the equation

$$\frac{d\phi}{dz} = \frac{(z_1z_2)^{\frac{1}{2}}}{2\,.\,a^{\frac{1}{2}}}\,\frac{(a+z)^{\frac{1}{2}}}{z}\,\frac{1}{[(z-z_1)(z_2-z)]^{\frac{1}{2}}},$$

which is, as is natural, an elliptic function. The amount by which ϕ increases as the particle moves from its highest to its lowest point

$$= \left(\frac{z_1z_2}{a}\right)^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\left(\frac{a+z_1\cos^2\theta+z_2\sin^2\theta\,d\theta}{z_1\cos^2\theta+z_2\sin^2\theta}\right)^{\frac{1}{2}},$$

and therefore lies between $\frac{1}{2}\pi\left(1+\frac{z_1}{a}\right)^{\frac{1}{2}}$ and $\frac{1}{2}\pi\left(1+\frac{z_2}{a}\right)^{\frac{1}{2}}$.

It will be found that this integral can be expressed in the form

$$\int_\alpha^\beta \frac{x\,dx}{2[x(x-1)(x-\alpha)(\beta-x)]^{\frac{1}{2}}}, \text{ where } \alpha = \frac{a+z_2}{z_2},\quad \beta = \frac{a+z_1}{z_1}.$$

6101. (By J. J. WALKER, M.A.)—If one side of a thin bi-convex lens is turned towards the sun, a well-defined image may be formed on a small screen facing that side, say at a distance of α^{-1}. The same being done with the other side, and the distance being now α'^{-1}, show that, if μ is the refractive index, the power of the lens is given by

$$2(2\mu-1)\phi = -(\mu-1)(\alpha+\alpha').$$

Solution by the PROPOSER.

The image referred to is formed by pencils which, undergoing refraction at the first surface of the lens, internal reflection at the opposite face, and

finally a second refraction on emergence from the first face, are brought to real foci in front of the lens. Let the reciprocals of the distances of the foci of the first refracted and reflected pencils from the lens be β, γ; the curvatures of the faces ρ, ρ' respectively, in the first position. Then

$$\mu\beta = (\mu-1)\rho, \quad \gamma+\beta = 2\rho', \quad \alpha-\mu\gamma = (1-\mu)\rho;$$

among which equations eliminating β, γ, $\alpha = -2(\mu-1)\rho + 2\mu\rho'$, an essentially positive expression, since ρ is negative, ρ' positive. When the lens is reversed, ρ and ρ' change signs, so that $\alpha' = 2(\mu-1)\rho' - 2\mu\rho$. Hence $\alpha+\alpha' = -2(2\mu-1)(\rho-\rho')$; but $\phi = (\mu-1)(\rho-\rho')$, and therefore $2(2\mu-1)\phi = -(\mu-1)(\alpha+\alpha')$.

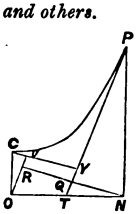
The values of α, α' show that there will be two real images formed by the meniscus if the curvatures are in a ratio of greater inequality than $\mu : \mu-1$; one by the convexo-concave lens if of less inequality, the other being virtual. Those formed by the bi-concave lens are both virtual.

6454. (By W. H. H. Hudson, M.A.)—If NQ, CY be the perpendiculars from N, the intersection of the ordinate and directrix, and from C, the vertex, on the tangent at any point P of a catenary, and O be the point where the directrix meets the axis, prove that

$$QY : OC = \widehat{CP} - ON : PN.$$

Solution by J. Hammond, M.A.; R. Leidhold, M.A.; *and others.*

The following are well-known properties of the catenary:



$$PN = OC\cosh\frac{ON}{OC}, \quad \widehat{CP} = OC\sinh\frac{ON}{OC},$$

$$\tan PTN = \sinh\frac{ON}{OC};$$

and the last may be written $\sec PTN = \cosh\dfrac{ON}{OC}$.

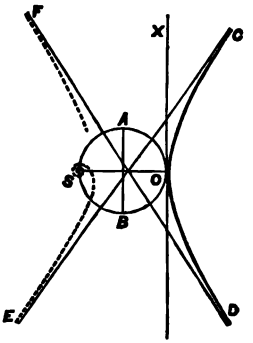
Now we have $QY = OC\sin PTN - ON\cos PTN$,

therefore $QY\cosh\dfrac{ON}{OC} = OC\sinh\dfrac{ON}{OC} - ON$, or $\dfrac{QY\,.\,PN}{OC} = \widehat{CP} - ON$.

[Otherwise: If ORV be parallel to PYQT, we have $OV : OC = PQ : PN$, and $OR : NQ = ON : PN$; but $OV - OR = QY$, $NQ = OC$, and $PQ = \widehat{CP}$; therefore $QY : OC = \widehat{CP} - ON : PN$.]

6460. (By J. W. Russell, M.A.)—Shew that, as the infinite branch of the curve $r = \dfrac{2a}{1-\theta^2}$ rolls on its initial tangent, the pole will describe a semicircle.

Solution by W. B. Grove, B.A.; C. Bickerdike, B.A.; *and others.*



As θ varies from -1 to 1, the portion DOC of the curve is described. Take OS, OX as axes. The locus of S, as DOC rolls on OX, is $y^2+(x-a)^2=a^2$,

that is, the circle SAOB.

By means of the equation $x=\frac{ar}{r-a}$, connecting the abscissa of S with the radius vector, we see that, when r is infinite, $x=a$; so that, as DOC rolls on OX, S describes the semicircle ASB.

6392. (By E. B. Elliott, M.A.)—A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2=x_1OA_1^2+x_2OA_2^2+x_3OA_3^2+x_4OA_4^2-\Sigma(a_{12}^2x_1x_2).$$

Solution by D. Edwardes; G. F. Walker, M.A.; *and others.*

Let (x', y', z', w') be the coordinates of O; then we have

$$-OP^2=\Sigma a_1a_2^2(x_1-x')(x_2-y') \quad \ldots\ldots(1),$$

and $$-OA_1^2=\Sigma a_1a_2^2x'y'-a_1a_2^2y'-a_1a_3^2z'-a_1a_4^2x'.$$

Hence the coefficient of x_1 in (1) is $-OA_1^2-\Sigma a_1a_2^2x'y'$,

so that $$-OP^2=-x_1OA_1^2-x_2OA_2^2-x_3OA_3^2-x_4OA_4^2+$$
$$\Sigma a_1a_2^2x_1x_2+\Sigma a_1a_2^2x'y'(1-x_1-x_2-x_3-x_4);$$

or, since $x_1+x_2+x_3+x_4=1$,

$$OP^2=x_1OA_1^2+x_2OA_2^2+x_3OA_3^2+x_4OA_4^2-\Sigma a_1a_2^2x_1x_2.$$

6526. (By Prof. Wolstenholme, M.A.)—Prove that, if p be positive and <1,

$$\int_0^1(x^p+x^{-p})\log(1+x)\frac{dx}{x}=\frac{\pi}{p\sin p\pi}-\frac{1}{p^2} \quad \ldots\ldots(1),$$

and

$$\int_0^1(x^p+x^{-p})\log(1-x)\frac{dx}{x}=\frac{\pi}{p}\cot p\pi-\frac{1}{p^2} \quad \ldots\ldots(2),$$

of which (1) may be deduced from (2) by putting x^2 for x.

Solution by Prof. NASH, M.A.; J. HAMMOND, M.A.; *and others.*

$$\int_0^1 (x^p + x^{-p}) \log(1-x) \frac{dx}{x} = -\int_0^1 (x^p + x^{-p}) \left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots\right) dx$$

$$= -\left[\frac{x^{p+1}}{p+1} + \frac{x^{p+2}}{2(p+2)} + \frac{x^{p+3}}{3(p+3)} + \ldots - \frac{x^{-p+1}}{p-1} - \frac{x^{-p+2}}{2(p-2)} - \ldots\right]_0^1$$

$$= \frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{2}\left(\frac{1}{p-2} - \frac{1}{p+2}\right) + \frac{1}{3}\left(\frac{1}{p-3} - \frac{1}{p+3}\right) + \&c.$$

$$= \frac{2}{p^2-1} + \frac{2}{p^2-2^2} + \frac{2}{p^2-3^2} + \&c.$$

$$= \frac{\pi}{p}\left\{\frac{2p\pi}{p^2\pi^2-\pi^2} + \frac{2p\pi}{p^2\pi^2-2^2\pi^2} + \frac{2p\pi}{p^2\pi^2-3^2\pi^2} + \&c.\right\}$$

$$= \frac{\pi}{p}\left\{\cot p\pi - \frac{1}{p\pi}\right\} = \frac{\pi}{p}\cot p\pi - \frac{1}{p^2}.$$

Putting x^2 for x in this integral, we have

$$\int_0^1 (x^{2p} + x^{-2p}) \log(1-x^2) \frac{2dx}{x} = 2\int_0^1 (x^{2p} + x^{-2p}) \log(1-x) \frac{dx}{x} + 2\int_0^1 (x^{2p} + x^{-2p}) \log(1+x) \frac{dx}{x};$$

$$\therefore \quad 2\int_0^1 (x^{2p} + x^{-2p}) \log(1+x) \frac{dx}{x} = \frac{\pi}{p}\cot p\pi - \frac{1}{p^2} - 2\left(\frac{\pi}{2p}\cot 2p\pi - \frac{1}{4p^2}\right)$$

$$= \frac{\pi}{p\sin 2p\pi} - \frac{1}{2p^2};$$

hence, writing p for $2p$, $\int_0^1 (x^p + x^{-p}) \log(1+x) \frac{dx}{x} = \frac{\pi}{p}\operatorname{cosec} \pi p - \frac{1}{p^2}$.

6401. (By R. KNOWLES, B.A., L.C.P.)—Prove that the locus of the middle points of normal chords to a parabola is the curve

$$(y^2 + 2a^2)^2 - 2axy^2 + 4a^4 = 0.$$

Solution by W. E. WRIGHT, B.A.; A. L. SELBY, B.A.; *and others.*

If (x', y') be one end of a normal chord, the other is

$$\frac{1}{4a}\left(\frac{8a^2}{y'} + y'\right)^2, \quad -\left(\frac{8a^2}{y'} + y'\right).$$

Let (x, y) be the middle of the chord, then we have

$$x = \frac{x'}{2} + \frac{1}{8a}\left(\frac{8a^2}{y'} + y'\right)^2, \quad y = \frac{y'}{2} - \frac{1}{2}\left(\frac{8a^2}{y'} + y'\right);$$

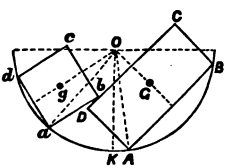
but $y'^2 = 4ax'$, therefore $x = x' + \frac{2a^2}{x'} + 2a$, and $y = -\frac{4a^2}{y'}$.

Using these values we obtain the stated equation.

4584. (By W. Siverley.)—Two uniform cylinders, of lengths $2a$ and $2b$, and the radius of each being r, rest in contact in a fixed smooth hemispherical bowl of radius R, their axes being in a vertical plane; find their position of equilibrium.

Solution by A. Martin, M.A.; *the* Proposer; *and others.*

Let ABCD be the larger cylinder, A and B touching the bowl at the lower and upper ends respectively, G its centre of gravity; *abcd*, *g* the like points in the other cylinder, *d* being the point of contact of the two cylinders, O the centre of the bowl; join OG, O*g*, cutting the cylinder in H and *h*, draw the radius OK vertical, and let $\angle GOK = \theta$, $\angle GOg = \beta$.



Taking moments about O, we have

$$2a\,[(R^2-a^2)^{\frac{1}{2}}-r]\sin\theta = 2b\,[(R^2-b^2)^{\frac{1}{2}}-r]\sin(\beta-\theta).$$

Whence $$\tan\theta = \frac{b\,[(R^2-b^2)^{\frac{1}{2}}-r]\sin\beta}{a\,[(R^2-a^2)^{\frac{1}{2}}-r] + b\,[(R^2-b^2)^{\frac{1}{2}}-r]\cos\beta}.$$

In the quadrilateral OH*dh*, we have

$$OH = (R^2-a^2)^{\frac{1}{2}}-2r, \quad Oh = (R^2-b^2)^{\frac{1}{2}}-2r, \quad hd = b,$$

and *h* and H are right angles; hence β may be determined.

6203. (By W. H. H. Hudson, M.A.)—If P be the population and W the wealth of an industrial community at a time t, (1) interpret $\frac{d}{dt}\left(\frac{W}{P}\right)$; and (2) show that, in such a community, if each person saves one-nth of his income, the average value of a person's work per unit of time is

$$n\frac{d}{dt}\left(\frac{W}{P}\right) + \frac{W}{P^2}\frac{dP}{dt}.$$

Solution by the Proposer.

$\frac{d}{dt}\left(\frac{W}{P}\right)$ is the average rate of increase (per unit of time) of a person's wealth, *i.e.*, the average person's rate of saving per unit of time; hence $\frac{d}{dt}\left(\frac{W}{P}\right)\delta t$ is the average person's saving in time δt, and $(n-1)\frac{d}{dt}\left(\frac{W}{P}\right)\delta t$ is the average person's expenditure in time δt. Now, let X be the average value of a person's work per unit of time, then $PX\delta t$ is the value of the whole community's work in time δt [or, strictly, the value is intermediate to $PX'\delta t$ and $(P+\delta P)X''\delta t$, where X', X'' are the least and greatest values that X can have during the interval δt, the values differing from

$PX\delta t$ by quantities of the second order]; therefore $PX\delta t - \delta W$ is the expenditure of the whole community in time δt

$$= P \text{ multiplied by } (n-1)\frac{d}{dt}\left(\frac{W}{P}\right)\delta t;$$

therefore $$X = \frac{1}{P}\frac{dW}{dt} + n\frac{d}{dt}\left(\frac{W}{P}\right) - \frac{d}{dt}\left(\frac{W}{P}\right) = n\frac{d}{dt}\left(\frac{W}{P}\right) + \frac{W}{P^2}\frac{dP}{dt}.$$

Otherwise :—Each person in time δt must do work sufficient to provide himself with an *income* equal to n times his saving, and also must perform his share of providing average *wealth* for the increment of population; therefore $$X\delta t = n\frac{d}{dt}\left(\frac{W}{P}\right)\delta t + \frac{1}{P}\cdot\frac{W}{P}\cdot\delta P.$$

6072. (By Prof. Cochez.)—Sur les côtés d'un triangle et dans le même sens on prend à partir des sommets de longueurs AD, BE, CF égales respectivement à $\frac{m\text{AB}}{n}$, $\frac{m\text{BC}}{n}$, $\frac{m\text{AC}}{n}$, on joint les points D, E, F. Déterminer (1) le rapport $\frac{\text{DEF}}{\text{ABC}}$; (2) le rapport $\frac{a^2+b^2+c^2}{a'^2+b'^2+c'^2}$, a, b, c étant les côtés de ABC, a', b', c' les côtés DEF; (3) Comment faut-il déterminer le rapport $\frac{m}{n}$ pour que le triangle DEF ait une surface minima ?

Solution by G. Heppel, M.A.; E. Rutter; *and others.*

Suppose that a', b', c' represent the sides FD, DE, EF respectively. Then, by equating the values of cos C, as determined from the triangles EFC and ABC, we obtain

$$n^2c'^2 = (2m^2 - 3mn + n^2)a^2 + (2m^2 - mn)b^2 - (m^2 - mn)c^2;$$

hence, by symmetry and addition,

$$a'^2 + b'^2 + c'^2 = \frac{n^2 - 3mn + 3m^2}{n^2}(a^2 + b^2 + c^2).$$

Also $$\frac{\text{FEC}}{\text{ABC}} = m(n-m); \quad \text{hence} \quad \frac{\text{DEF}}{\text{ABC}} = \frac{n^2 - 3mn + 3m^2}{n^2},$$

the same ratio as before. This may be written

$$\frac{1}{4} + 3\left(\frac{m}{n} - \frac{1}{2}\right)^2; \quad \text{therefore DEF is a minimum when } \frac{m}{n} = \frac{1}{2}.$$

6089. (By D. Edwardes.)—If A, B, C are the angles of a triangle, prove that

$$2\{\sin^2(A-B) + \sin^2(B-C) + \sin^2(C-A) + \sin(A-B)\sin(C-A)\cos A + \sin(B-C)\sin(A-B)\cos B + \sin(C-A)\sin(B-C)\cos C\}$$
$$= (\sin^2 A + \sin^2 B + \sin^2 C)(1 - 8\cos A\cos B\cos C).$$

Solution by G. Heppel, M.A.; W. B. Grove, B.A.; *and others.*

$$2\sin^2(A-B)+2\sin^2(A+B)=4\sin^2A+4\sin^2B-8\sin^2A\sin^2B;$$

$$\therefore\quad U=2\sin^2(A-B)=4\sin^2A+4\sin^2B-2\sin^2C-8\sin^2A\sin^2B,$$

$$V=2\sin(C-A)\sin(B-C)\cos C=\sin(C-A)[\sin B+\sin(B-2C)]$$
$$=\sin^2C-\sin^2A-\tfrac{1}{2}\cos 4C+\tfrac{1}{2}\cos 2B$$
$$=\sin^2C-\sin^2A-\sin^2B+4\sin^2C-4\sin^4C;$$

therefore $U+V=3\sin^2A+3\sin^2B+3\sin^2C-8\sin^2A\sin^2B-4\sin^4C;$

$\therefore$ whole expression $=(\sin^2A+\sin^2B+\sin^2C)[9-4(\sin^2A+\sin^2B+\sin^2C)]$
$=(\sin^2A+\sin^2B+\sin^2C)(1-8\cos A\cos B\cos C).$

6415. (By C. W. Merrifield, F.R.S.)—A regular skeleton dodecahedron is made of 30 equal wires connected 3 by 3 with universal joints. It is then flattened symmetrically, so that two opposite pentagonal faces retain their shape—these two faces being connected with the corners of a regular decagon by 10 of the edges. Show that, in order to make the construction possible, each pentagon must be turned through $\frac{1}{2}\tan^{-1}2$.

I. *Solution by* C. A. Swift, B.A.; G. F. Walker, M.A.; *and others.*

The angle turned through will be the vertical angle of a triangle whose base is one of the edges and sides are the radii of the circles circumscribing a pentagon and a decagon having the same sides.

If r_1, r_2 be these radii, a a side, and θ the required angle, we have

$$r_1=a.\frac{5^{\frac{1}{2}}+1}{2},\quad r_2=a.\frac{(5^{\frac{1}{2}}+1)}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}};$$

$$a^2=a^2\left(\frac{5^{\frac{1}{2}}+1}{2}\right)^2+\frac{5^{\frac{1}{2}}+1}{2.5^{\frac{1}{2}}}a^2-2a^2.\frac{5^{\frac{1}{2}}+1}{2}.\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.2^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta,$$

$$1=\frac{6+2.5^{\frac{1}{2}}}{4}+\frac{5+5^{\frac{1}{2}}}{10}-\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta;$$

$$\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta=\frac{60+20.5^{\frac{1}{2}}+20+4.5^{\frac{1}{2}}-40}{40}=\frac{6+2.5^{\frac{1}{2}}}{2.5^{\frac{1}{2}}};$$

therefore $\cos\theta=\dfrac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})},\quad \cos 2\theta=\dfrac{5^{\frac{1}{2}}+1}{5^{\frac{1}{2}}}-1=\dfrac{1}{5^{\frac{1}{2}}};$

and $\tan 2\theta=2,\quad \theta=\frac{1}{2}\tan^{-1}2.$

II. *Solution by the* Proposer.

Let A be the centre of figure of the two pentagons and the decagon. Now the figure must be twisted, so that the radii of the circles circumscribed to the pentagon and decagon may form a triangle of which the

third side is an edge of the solid; that is to say, if B is a corner of the pentagon, and C of the decagon, ABC will be a triangle of which

$$AB = \frac{1}{2\sin 36^\circ},\quad BC = 1,\quad CA = \frac{1}{2\sin 18^\circ};$$

and A is the half-angle of revolution; for, if there were no twist, ABC would be in a straight line.

Now, $\cos A = \dfrac{AB^2+AC^2-BC^2}{2AB\,.\,AC} = \dfrac{\sin^2 18^\circ+\sin^2 36^\circ-4\sin^2 18^\circ\sin^2 36^\circ}{2\sin 18^\circ\sin 36^\circ}$;

substituting the values of these sines, we obtain

$$\cos A = \frac{\sqrt{5}-1}{2\sqrt{(5-2\sqrt{5})}},\quad \cos 2A = \frac{1}{\sqrt{5}},\quad \tan 2A = 2.$$

Hence, the angle turned through by one pentagon upon the other, supposing them to turn different ways, is arc tan 2. If they both turn one way, they would turn about the same angle; but, in either case, the angle A, which is the twist of each relatively to the decagon, would remain as above, namely ½ arc tan A.

It does not appear that the skeleton dodecahedron can be flattened symmetrically about an axis joining two opposite corners.

6145. (By C. Leudesdorf, M.A.)—Find three square numbers such that, if each be subtracted from the sum of the other two, and the remainders doubled, the resulting numbers shall be perfect squares.

I. *Solution by* G. Heppel, M.A.; S. Tebay, B.A.; *and others.*

This is obviously the same thing as requiring the solution in whole numbers of the equations $x^2+y^2 = w^2$, $y^2+z^2 = u^2$, $z^2+x^2 = v^2$, where u^2, v^2, w^2 are the numbers sought.

If $y = mx$ and $z = nx$, these equations are the same as

$$1+m^2 = \square\ ;\quad 1+n^2 = \square\ ;\quad m^2+n^2 = \square \ldots\ldots\ldots (1, 2, 3).$$

From (1), (2), $m = \dfrac{1-h^2}{2h}$, $n = \dfrac{1-k^2}{2k}$; and from (3), $\dfrac{m}{n} = \dfrac{1-r^2}{2r}$;

therefore $$\frac{k(1-h^2)}{h(1-k^2)} = \frac{1-r^2}{2r};$$

and solving this as a quadratic to find r, we see that $h^2k^2(h^2+k^2)-4h^2k^2+h^2+k^2$ must be a square, a condition which is satisfied if $h^2+k^2 = 4$; and it is easy to show that in that case we have

$$h = \frac{4l}{l^2+1},\qquad k = \frac{2(l^2-1)}{l^2+1}.$$

Assigning any value for l, we obtain h, k, m, n, x, y, z, u, v, w.

For example, if $l = 3$; $h = \frac{6}{5}$; $k = \frac{8}{5}$; $m = -\frac{1}{60}$; $n = -\frac{32}{60}$; and, taking the least integral values of the rest, we have

$$x = 240,\quad y = 44,\quad z = 117,\quad u = 125,\quad v = 267,\quad w = 244.$$

II. *Solution by the* PROPOSER.

If x^2, y^2, z^2 are the numbers, $2(y^2+z^2-x^2) = a^2$, $2(z^2+x^2-y^2) = b^2$, $2(x^2+y^2-z^2) = c^2$; or, $a^2+4x^2 = b^2+4y^2 = c^2+4z^2 = a^2+b^2+c^2$.

These equations will be satisfied by

$$2x = 2z\cos\alpha + c\sin\alpha, \quad a = 2z\sin\alpha - c\cos\alpha, \quad 2y = 2z\cos\beta + c\sin\beta,$$
$$b = 2z\sin\beta - c\cos\beta, \quad 4z^2 = (2z\sin\alpha - c\cos\alpha)^2 + (2z\sin\beta - c\cos\beta)^2.$$

Let $\beta = \frac{1}{2}\pi - \alpha$; then the last equation gives $z = \frac{1}{4}c\operatorname{cosec} 2\alpha$, whence $x = \frac{1}{2}c(\frac{1}{4}\operatorname{cosec}\alpha + \sin\alpha)$ and $y = \frac{1}{2}c(\frac{1}{4}\sec\alpha + \cos\alpha)$; or if $\cot\frac{1}{2}\alpha = mn^{-1}$,

$$x = \tfrac{1}{2}c\left\{\frac{m^2+n^2}{8mn} + \frac{2mn}{m^2+n^2}\right\}, \quad y = \tfrac{1}{2}c\left\{\frac{m^2+n^2}{4(m^2-n^2)} + \frac{m^2-n^2}{m^2+n^2}\right\},$$
$$z = \frac{c}{16}\,\frac{(m^2+n^2)^2}{mn(m^2-n^2)}.$$

For integral values, put $c = 16mn(m^4-n^4)$; then

$$x = (m^2-n^2)(m^4+n^4+18m^2n^2), \quad y = 2mn(5m^4+5n^4-6m^2n^2),$$
$$z = (m^2+n^2)^3,$$

$m = 2$, $n = 1$ give $z = 125$, $y = 244$, $x = 267$.

6292 & 6325. (By J. McDOWELL, M.A., and R. KNOWLES, B.A., L.C.P.)—If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \ldots + \frac{1}{n}(-\tfrac{1}{2})^{n-1}p_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} \quad \ldots\ldots(1),$$
$$p_1 - 2p_2 + \ldots + (n-1)(-)^{n-2}p_{n-1} = 0 \quad \ldots\ldots(2),$$
$$\tfrac{1}{2}p_1 - \tfrac{1}{3}p_2 + \ldots + \frac{1}{n+1}(-)^{n-1}p_n = \frac{n}{n+1} \quad \ldots\ldots(3).$$

Solution by the Rev. D. THOMAS, M.A.; G. HEPPEL, M.A.; *and others.*

(1) Let s_n denote the series on the left-hand side of the equation; then

$$n(s_n - s_{n-1}) = 1-(1-1)^n = 1, \text{ therefore } s_n = s_{n-1} + \frac{1}{n};$$

therefore
$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}.$$

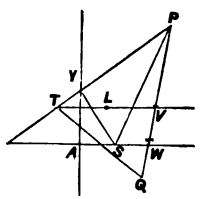
(2) If C_r^n denote the number of combinations of n things taken r at a time, $\Sigma_{r=1}^{r=n}(-)^{r-1}C_r^n = n\Sigma_{r=1}^{r=n}(-)^{r-1}C_{r-1}^{n-1} = n(1-1)^{n-1} = 0$.

(3) $\Sigma_{r=1}^{r=n}(-)^{r-1}\dfrac{C_r^n}{r+1} = \dfrac{1}{n+1}\Sigma_{r=1}^{r=n}(-)^{r-1}C_{r+1}^{n+1}$

$$= \frac{1}{n+1}\left\{\frac{(n+1)n}{2!} - \frac{(n+1)n(n-1)}{3!} + \ldots + (-)^{n+1}\right\} = \frac{n}{n+1}.$$

6429. (By R. Tucker, M.A.)—Given two tangents to a parabola, and the point on the axis through which the chord of contact passes; construct the parabola.

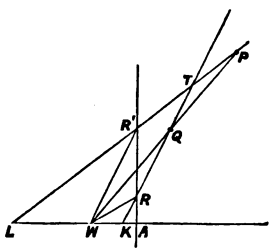
I. *Solution by* R. E. Riley, B.A.; R. Knowles, B.A., L.C.P.; *and others.*



Let TP, TQ be the given tangents; PQ the chord of contact; and W the point on the axis. Bisect PQ at V; then TV is a diameter. Draw SW parallel to TV; then SW is the axis. Make the angle TPS = PTV; then S is the focus; and if we draw SY perpendicular to TP, and through Y draw YA at right angles to axis, A will be the vertex, and the curve is determined.

The line PS is fixed in direction; and if L be the middle point of TV, L is the vertex of diameter TV; hence $4LS \cdot LV = PV^2$. Now, LV and PV being constant, LS is constant, and it seems that there would be only two positions of S, and therefore of A.

II. *Solution by the* Proposer.



The equation to PQ through P (m_1), Q (m_2) is $2m_1m_2x - y(m_1 + m_2) + 2a = 0$;

therefore $AK = -\frac{a}{m_1 m_2}$.

The equation to WR (parallel to PT) is

$$y = m_1\left(x + \frac{a}{m_1 m_2}\right),$$

therefore $AR = \frac{a}{m_2}$;

that is, R is the point on the tangent at the vertex where QT meets that line.

Hence, the construction is as follows:—through W draw straight lines parallel to the given tangents, meeting them in R, R'; RR' is the tangent at the vertex, whence the rest obviously follows.

6356. (By C. Leudesdorf, M.A.)—If 7 be divided by 5, giving 1 and 2 over; and then successively 21 by 5, giving 4 and 1 over; 14 by 5, giving 2 and 4 over; 42 by 5, giving 8 and 2 over, and so on until the figures recur, the result of placing all the quotients after the 7 and adding a decimal point suitably is $\cdot 714285$, which is equal to $\frac{5}{7}$. Investigate the general form of the fractions $a : b$ which can be expressed as circulating decimals by dividing b by a in the manner explained with reference to the case of $\frac{5}{7}$.

I. *Solution by* W. H. WALENN; C. HARKEMA; *and others.*

Proposition I.—Decimal equivalents of all reciprocals of the form $(10a-1)^{-1}$ have the terminal figure of their period equal to 1. For, in all cases, the dividend that yields the first recurring figure is the same as that which commences the work; in the present instance 10. To effect this the remainder of the previous division must be 1. In $(10a-1)^{-1}$, the quotient figure which yields one for a remainder is always 1. Therefore 1 is the last figure of the period.

Proposition II.—The decimal equivalents of reciprocals of the form $(10a-1)^{-1}$ may be constructed by means of the multiplier a, this multiplier being used to obtain the next figure towards the left from the previous one as a multiplicand, and adding in the tens' digit that may be carried from the previous product.

For, by actual division,

$$(10a-1)^{-1} = (10a)^{-1} + (10a)^{-2} + \ldots + (10a)^{-n+1} + (10a)^{-n},$$

n being the number of figures in the recurring period of the decimal, and by Proposition I., $(10a)^{-n} = 1$, consequently $(10a)^{-n+1} = 10a$, and the series (commencing from the right-hand term) becomes

$$\ldots + (10a)^2 + (10a)^2 + 10a + 1.$$

By exactly similar reasoning this proposition or theorem may be extended to reciprocals having more than two figures in the denominator.

Proposition III.—Reciprocals of the form $(10a-3)^{-1}$ have 7 for the last figure of their period.

Since $7^2 = 49$, the multiplier which will reduce these reciprocals to a multiple of $(10a-1)^{-1}$ is 7. Therefore

(since $7 \cdot 1 = 7$), 7 is the last figure of the period.

In a similar way (since $3^2 = 9$), $(10a-7)^{-1}$ has 3 for the last figure of its period, and $(10a-9)^{-1}$ has 9 for the last figure of its period.

The modified multiplication treated of in Proposition II. will be found to be the inverse operation to the modified division in the question, by comparing the multiplicands, products, and carrying figures of the one with the quotients, dividends, and remainders of the other process.

The value of a in Proposition II. is also the value of the divisor referred to in the question.

In the case of $\frac{1}{7}$, the last figure is 5, since $\frac{1}{7} = 5 \cdot \frac{1}{7}$, and $7 \cdot 5 = 35$, the last figure of which is 5.

II. *Solution by the* PROPOSER.

Let the quotients be $q_1, q_2, \ldots$, the remainders $r_1, r_2, \ldots$, and the resulting decimal

$$D = \frac{b}{10} + \frac{q_1}{10^2} + \frac{q_2}{10^3} + \ldots;$$

then $b = aq_1 + r_1$, $10r_1 + q_1 = aq_2 + r_2$, $10r_2 + q_2 = aq_3 + r_3$, &c.;

therefore $$\frac{b}{10} + \frac{10r_1 + q_1}{10^2} + \ldots = \frac{aq_1 + r_1}{10} + \frac{aq_2 + r_2}{10^2} + \ldots;$$

therefore $$\frac{b}{10} + \frac{q_1}{10^2} + \ldots = \frac{aq_1}{10} + \frac{aq_2}{10^2} + \ldots,$$

$$D = a(10D - b), \quad D = \frac{ab}{10a-1}.$$

Now, if $D = \frac{a}{b}$, $a = \frac{b^2+1}{10}$, and b must be of one of the forms $10n+3$, $10n+7$, which gives $a = 10n^2+6n+1$ or $10n^2+14n+5$. Thus the general forms of the fractions $\frac{a}{b}$, which possess the property enunciated, are

$$\frac{10n^2+6n+1}{10n+3} \text{ and } \frac{10n^2+14n+5}{10n+7},$$

where n is any integer.

6218. (By Professor SYLVESTER, F.R.S.)—If

$$E = a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e \ldots, \quad F = a\delta_c + 3b\delta_d + 6c\delta_e + 10d\delta_f \ldots,$$
$$G = a\delta_d + 4b\delta_e + 10c\delta_f + 20d\delta_g \ldots, \quad H = a\delta_e + 5b\delta_f + 15c\delta_g + 35e\delta_h \ldots,$$

express $(E)^n$ as an algebraical function of E, F, G, H, &c.

Solution by W. J. CURRAN SHARP, M.A.

Let $E = E' + E''$, where E' applies only to the function operated upon, and E'' only to the a, b, &c. involved in the operative symbols; then

$$E'E \equiv F, \quad (E'')^2E \equiv E''F \equiv G, \quad (E'')^3E \equiv (E'')^2F \equiv E''G \equiv H + \&c.;$$

therefore

$$(E)^2 = (E' + E'')\,E = (E')^2 + F,$$
$$(E)^3 = (E' + E'')\,[(E')^2 + F] = (E')^3 + 3E'F + G,$$
$$(E)^4 = (E')^4 + 6\,(E')^2F + 4E'G + H, \ \ldots,$$
$$(E)^n = (E')^n + \frac{\mu(\mu-1)}{1.2}(E')^{n-2}F + \frac{\mu(\mu-1)(\mu-2)}{1.2.3}(E')^{n-3}G + \&c.$$

From these equations the values of $(E')^2$, $(E')^3$, &c., in terms of E, F, &c., may be successively determined, the first results being

$$(E')^2 = (E)^2 - F, \quad (E')^3 = (E)^3 - 3EF + 2G,$$
$$(E')^4 = (E)^4 - 6\,(E)^2F + 6F^2 - 4EG + 3H, \text{ &c.}$$

6008. (By Professor SYLVESTER, F.R.S.)—Prove that the equation

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = 0$$

is insoluble by positive or negative integer values of x, y, z.

Solution by W. J. CURRAN SHARP, M.A.

If

$$\frac{y}{z} + \frac{z}{y} = p, \quad \frac{z}{x} + \frac{x}{z} = q, \quad \frac{x}{y} + \frac{y}{x} = r,$$

p, q, r are each numerically > 2; and $p + q + r = 0$ by the given equation; also, by elimination, $pqr + 4 = p^2 + q^2 + r^2$; therefore $pqr + 4 > 0$, and con-

sequently pqr is positive; and hence two of the quantities p, q, r must be negative, and only one of x, y, z is negative. Let this be z; then, from the given equation, we have

$$\left(\frac{xy}{z^2}+1\right)(x+y) = -\frac{1}{z}(x^2+y^2) < -\frac{2xy}{z}, \qquad \left(\frac{xy}{z^2}+1\right)(x+y)+\frac{2xy}{z} < 0,$$

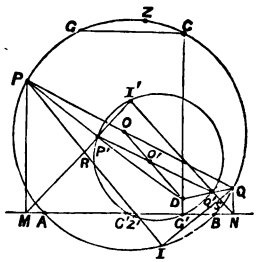
or

$$\left(\frac{x^2}{z^2}+\frac{2x}{z}+1\right)y+\frac{xy^2}{z^2}+x < 0,$$

which is impossible on the suppositions. Hence the equation is insoluble, &c.

[Prof. SYLVESTER states that this method of proof is quite distinct from his own, which proceeds upon the usual plan of showing that if the equation, or rather, a certain allied equation, is true for a given set of numbers, it will be true for another set with an inferior dominant (the dominant of a set of numbers meaning the greatest one of them without respect to sign). He believes Mr. SHARP's proof to be the first instance in questions of this kind where a direct method of negation has been employed, and if so, it may be regarded as constituting a new departure in the subject.]

6290. (See p. 38). II. *Solution by* Prof. TOWNSEND F.R.S.



Let A, B, C be the three vertices of the triangle; A′, B′, C′ the three middle points of its opposite sides; D its orthocentre; E, F, G the three second intersections with its circumscribing circle of the three parallels through A, B, C to its opposite sides; E′, F′, G′ the feet of three perpendiculars from A, B, C to its opposite sides; X, Y, Z the three points of trisection nearest to A, B, C of the three arcs AE, BF, CG of its circumscribing circle; X′, Y′, Z′ the three nearest to A′, B′, C′ of the three arcs A′E′, B′F′, C′G′ of its nine-point circle; O and O′ the centres of the circumscribing and nine-point circles respectively; P and Q any pair of diametrically opposite points on the circumscribing circle, P′ and Q′ the corresponding pair bisecting PD and QD, and of course also diametrically opposite, on the nine-point circle; PM and QN the two perpendiculars from P and Q upon the side AB of the triangle; PR and QS the two from the same upon the two lines of connection MI′ and NQ′ of M and N with P′ and Q′ respectively; I the intersection of PR and QS, and I′ that of P′M and Q′N;—then, since (*Reprint*, Vol. IV., p. 17) MP′ and NQ′ are the two tangents at the vertices of the two parabolas of the system of which P and Q are the foci, therefore PR and QS are the two axes of the same two parabolas; and since (see same) the two former MP′ and NQ′ intersect at right angles at the point I′ on the nine-point circle, and are tangents to the tricuspidal hypocycloid circumscribed to that circle at the three points X′, Y′, Z′ of its circumference, therefore the two latter PR and QS, being consequently the two parallels to NQ′ and MP′ at the doubles of their two distances in the two opposite directions from the external centre of similitude D of the two

circles, intersect at right angles at the point I on the circumscribing circle, and are tangents to the tricuspidal hypocycloid circumscribed to that circle at the three points X, Y, Z of its circumference; and therefore &c., as regards the envelop in question.

6158. (By Professor CROFTON, F.R.S.)—Find the mean distance of two points in a rectangle.

Solution by C. J. MONRO, M.A.

$$\frac{1}{a^2b^2}\int_0^a d\xi \int_0^a d\xi' \int_0^b d\eta \int_0^b d\eta' \,.\, [(\xi'-\xi)^2+(\eta'-\eta)^2]^{\frac{1}{2}n}, \;=\; \mathrm{M}_n \text{ say},$$

will give the mean n^{th} power of the distance. It is enough to say that, if we distinguish between opposite directions, the expression for a finite *number* of distances will be replaced, on passing to infinity, by

$$\frac{a}{d\xi}\cdot\frac{a}{d\xi'}\cdot\frac{b}{d\eta}\cdot\frac{b}{d\eta'},$$

and the expression for the sum of their n^{th} powers by the rest of the compound symbol.

The variables are present only by their differences $\xi'-\xi$ and $\eta'-\eta$, say x and y. Therefore we may eliminate ξ' and η', and, independently of the finite factors, integrate immediately as to ξ and η. For, while ξ is constant, $d\xi' = dx$; the limits of x are $\pm a$, but the finite factors being an even function, we may take the limits 0 and a and double the integral; the lower limit of ξ is the higher value of the two 0 and $-x$, namely 0, and its higher limit, the lower value of the two a and $a-x$, namely $a-x$. The like is true of η', η, and y. Integrating, then, as to ξ and η, and doubling twice for x and y, we have

$$\mathrm{M}_n = \frac{4}{a^2b^2}\int_0^a dx \int_0^b dy \,.\, (a-x)\,(b-y)\,(x^2+y^2)^{\frac{1}{2}n}.$$

When n is even and not negative, the integral is algebraical and rational. In general, if $x^2+y^2 = r^2$, and

$$\mathrm{R} = abr^{n+1} - (b\cos\theta + a\sin\theta)\, r^{n+2} + \cos\theta\sin\theta\, r^{n+3},$$

and $\alpha+\beta$ being $\frac{1}{2}\pi$, $a\sec\beta = b\sec\alpha$, $= (a^2+b^2)^{\frac{1}{2}}$, the result may be written

$$\mathrm{M}_n = \frac{4}{a^2b^2}\int_0^\beta d\theta \int_0^{a\sec\theta} \mathrm{R}dr + \frac{4}{a^2b^2}\int_0^\alpha d\theta \int_0^{b\sec\theta} \mathrm{R}dr \ldots\ldots\ldots\ldots(1).$$

The former term is $\dfrac{4}{(n+3)\,a^2b^2}\displaystyle\int_0^\beta \left(\frac{a^{n+3}b}{n+2}\sec\theta^{n+2} - \frac{a^{n+4}}{n+4}\sec\theta^{n+3}\sin\theta\right) d\theta.$

If n is even or less than -1, this is a purely trigonometrical function; but if n is odd and not less than -1, it involves on reduction

$$\int_0^\beta \sec\theta\, d\theta, = \log \frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}, \quad \text{say } G\beta.$$

When, as proposed, $n = 1$, we have

$$\frac{1}{a^2b^2}\left[\frac{a^4b}{3\,.\,2}(\sec^2\beta\sin\beta+G\beta)-\frac{a^5}{5\,.\,3}(\sec^3\beta-1)\right],$$

which, if we put c^2 for a^2+b^2, is $\frac{c}{6}+\frac{a^2}{6b}G\beta-\frac{a^2c^3-a^5}{15a^2b^2}$. The latter term of (1) is got by interchanging b and β with a and α; and, adding the two, we get $M_1 = \frac{c}{3}+\frac{1}{6}\left(\frac{a^2}{b}G\beta+\frac{b^2}{a}G\alpha\right)-\frac{c^5-a^5-b^5}{15a^2b^2}$.

When a and α, or b and $\beta = 0$, the latter two terms destroy each other.

[See Williamson's *Integral Calculus*, p. 348.]

6377. (By Professor Genese, M.A.)—A quadrilateral linkage is capable in any position of having a circle inscribed in it. If one side be fixed, find the locus of the centre of the inscribed circle.

Solution by G. Heppel, M.A.; W. B. Grove, B.A.; *and others.*

Let AB be the fixed side, $AD = b$, $BC = c$; and let the angles at A and B be 2θ and 2ϕ. Then, obviously, $CD = d = b+c-a$. Taking A as origin, the point D is $x = b\cos 2\theta$, $y = b\sin 2\theta$, and the point C is $x = a-c\cos 2\phi$, $y = c\sin 2\phi$; hence we have

$$d^2 = (b\cos 2\theta + c\cos 2\phi - a)^2 + (b\sin 2\theta - c\sin 2\phi)^2 = (b+c-a)^2,$$

or
$$ab\sin^2\theta + ac\sin^2\phi = bc\sin^2(\theta+\phi).$$

If the centre of inscribed circle is the point (x, y),

$$\sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}},\ \sin\phi = \frac{y}{[(a-x)^2+y]^{\frac{1}{2}}},\ \sin(\theta+\phi) = \frac{ay}{(x^2+y^2)^{\frac{1}{2}}[(a-x)^2+y^2]}.$$

Substituting and reducing, we obtain, for the locus of the centre, the circle
$$\left(x-\frac{ab}{b+c}\right)^2+y^2 = \frac{abcd}{(b+c)^2}.$$

5452. (By Professor Sylvester, F.R.S.)—Show that the fact of an equation $fx = 0$ of the n^{th} degree having r pairs of roots such that the product of each pair is a given constant, is conditioned by equating to zero r functions each of the $(n-2r+1)^{\text{th}}$ order in the constants; and, as a corollary to the case of $r = 1$, show that the resultant of $f(x, y)$ and $f(y, x)$ is a perfect square multiplied by the product of two linear factors.

Solution by W. J. CURRAN SHARP, M.A.

If $f(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ be an equation which has r pairs of roots, such that the product of each pair $= c$. Then, by putting $x = c^{\frac{1}{2}} y$, it follows that the equation

$$c^{\frac{1}{2}n} y^n + p_1 c^{\frac{1}{2}(n-1)} y^{n-1} + \ldots + p_n = 0 \quad \ldots\ldots\ldots\ldots (1)$$

has r pairs of roots, such that the product of each pair is unity.

Let $y^{2r} + a_1 y^{2r-1} + a_2 y^{2r-2} + \ldots + a_2 y^2 + a_1 y + 1 = 0 \ldots\ldots\ldots\ldots (2)$

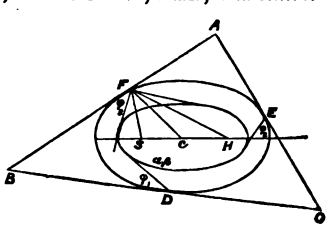
be the equation which gives these roots. The expression in (1) is divisible by that in (2), and therefore the remainder is 0 identically. This gives $2r$ equations involving the r quantities a_1, a_2, &c. to $n - 2r + 1^{\text{th}}$ power, and p_1, p_2, &c. to the first; and hence the resultants will be r functions of the $n - 2r + 1^{\text{th}}$ order in p_1, p_2, &c.

If $f(x, y) = 0$ and $f(y, x) = 0$ have a common root, the equation $f(x, 1) = 0$ must have a pair of roots whose product is unity, and the resultant must contain the function obtained above as a factor; it must also contain it again since $f(1, x) = 0$ has a similar pair, and this, with the product of p_0 and p_n, will produce the resultant.

6374. (By Professor TOWNSEND, F.R.S.)—A conic being supposed to touch the three sides of a triangle, show that the three pairs of tangents to any confocal conic from the three points of contact are three pairs of common tangents to three circles of corresponding radii having their centres at the vertices of the triangle.

I. *Solution by* G. F. WALKER, M.A.; W. B. GROVE, B.A.; *and others.*

Let ab, $\alpha\beta$ be the semi-axes of the given and confocal conics; $d_1 d_2 d_3$ the diameters parallel to BC, CA, AB; then



$$SF^2 + HF^2 + 2SF \cdot HF \cos 2\phi_3 = 4a^2,$$

$$2(CF^2 + CS^2) + 2d_3^2(1 - 2\sin^2\phi_3) = 4a^2,$$

$$a^2 + b^2 - (\alpha^2 + \beta^2) = 2d_3^2 \sin^2\phi$$

$$= 2d_1^2 \sin^2\phi_1 = 2d_2^2 \sin^2\phi_2;$$

$$\therefore \quad d_3 \sin\phi_3 = d_1 \sin\phi_1 = d_2 \sin\phi_2;$$

$$\frac{d_3}{d_1} = \frac{BF}{BD}; \quad \text{therefore} \quad BF \sin\phi_3 = BD \sin\phi_1;$$

and similarly for others. $BF \sin\phi_3$ is perpendicular on tangent from B; therefore, &c.

II. *Solution by the* PROPOSER.

Denoting by α, β, γ the three angles made with the three sides a, b, c of the triangle by the three pairs of corresponding tangents to any conic S of

the confocal system, and by α', β', γ' the three for any other conic S′ of the system; then, since, by a known property of confocal conics, $\sin\alpha : \sin\beta : \sin\gamma = \sin\alpha' : \sin\beta' : \sin\gamma'$, it follows at once that, if the property be true for any one conic S of the system, it is true for every other conic S′ of the system; but it is evidently true for the point-pair consisting of the two foci E and F of the system; and therefore &c.

6422. (By E. B. Elliott, M.A.)—Prove that the envelop of the line

$$x\cos\phi + y\sin\phi = ae^{\phi} + be^{-\phi},$$

where ϕ is a variable parameter, is the evolute of the evolute of the evolute of its own evolute.

Solution by the Proposer; Prof. Matz, M.A.; *and others.*

$$\varpi \equiv x\cos\phi + y\sin\phi - ae^{\phi} - be^{-\phi} = 0$$

being the type tangent, $\frac{d\varpi}{d\phi} = 0$ is the type normal, that is, the type tangent of the evolute. So $\frac{d^2\varpi}{d\phi^2}, \frac{d^3\varpi}{d\phi^3}, \ldots$ equated to zero, give the type tangents of the second, third, and successive evolutes. Now

$$\frac{d^4\varpi}{d\phi^4} \equiv x\cos\phi + y\sin\phi - ae^{\phi} - be^{-\phi} \equiv \varpi \equiv \frac{d^8\varpi}{d\phi^8} \equiv \frac{d^{12}\varpi}{d\phi^{12}} \equiv \ldots.$$

Thus the type tangent of the 4^{th}, 8^{th}, … $4n^{\text{th}}$ evolute is identical with that of the original curve. Wherefore, &c.

6186. (By T. R. Terry, M.A.)—If $a < 1$, and m and n are any positive integers, prove that

$$1 + \frac{(m-1)m}{1.n+1}\frac{a^2}{1-a^2} + \frac{(m-2)(m-1)m(m+1)}{1.2.(n+1)(n+2)}\frac{a^4}{(1-a^2)^2} + \ldots$$

$$= \frac{1}{(1-a^2)^{m-1}} \times \left\{1 - \frac{n-m+1}{n+1}(m-1)a^2 + \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)}\cdot\frac{(m-1)(m-2)}{1.2}a^4 - \ldots\right\}.$$

Solution by J. Hammond, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Suppose, more generally, that m and n are any quantities whatever, and let $S = 1 + \frac{(m-1)m}{1.n+1}\frac{a^2}{1-a^2} + \frac{(m-2)(m-1)m(m+1)}{1.2.(n+1)(n+2)}\frac{a^4}{(1-a^2)^2} + \ldots;$

then $$S = \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)}$$

$$\times \left\{ \frac{\Gamma(n-m+1)\,\Gamma(m)}{\Gamma(n+1)} + \frac{m-1}{1}\frac{\Gamma(n-m+1)\,\Gamma(m+1)}{\Gamma(n+2)}\frac{a^2}{1-a^2} \right.$$

$$\left. + \frac{(m-1)(m-2)}{1\,.\,2}\frac{\Gamma(n-m+1)\,\Gamma(m+2)}{\Gamma(n+3)}\frac{a^4}{(1-a^2)^2} + \&c. \right\}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)}\int_0^1 (1-x)^{n-m}x^{m-1}\left(1+\frac{a^2}{1-a^2}x\right)^{m-1}dx;$$

or, changing x into $1-x$,

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)}\int_0^1 (1-x)^{m-1}x^{n-m}\left(\frac{1-a^2x}{1-a^2}\right)^{m-1}dx$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)}\frac{1}{(1-a^2)^{m-1}}\left\{ \frac{\Gamma(m)\,\Gamma(n-m+1)}{\Gamma(n+1)} - \frac{\Gamma(m)\,\Gamma(n-m+2)}{\Gamma(n+2)} \right.$$

$$\left. \times (m-1)a^2 + \frac{\Gamma(m)\,\Gamma(n-m+3)}{\Gamma(n+3)}\frac{(m-1)(m-2)}{1\,.\,2}a^4 - \&c. \right\}$$

$$= \frac{1}{(1-a^2)^{m-1}}\left\{ 1 - \frac{n-m+1}{n+1}(m-1)a^2 + \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)} \right.$$

$$\left. \times \frac{(m-1)(m-2)}{1\,.\,2}a^4 - \&c. \right\}.$$

In the case where $n = 0$, $m = \frac{1}{2}$, we have from the definite integral, by an easy reduction, $$S = \frac{2(1-a^2)^{\frac{1}{2}}}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-a^2\sin^2\theta)^{\frac{1}{2}}}.$$

The final value of S is equivalent to the ordinary expansion, of this integral; the original value of S gives another expansion, which is only convergent when $a^2 < 1-a^2$ or $a^2 < \frac{1}{2}$.

[Mr. Terry remarks that this identity presented itself naturally in the consideration of the definite integral $\int_0^{\pi}\frac{\sin^{2n}x\,dx}{(1-2a\cos x+a^2)^m}$; but it admits however of the following simple proof:—

Let $$F\{\alpha, \beta, \gamma, x\} = 1 + \frac{\alpha\,.\,\beta}{\gamma\,.\,1}x + \frac{\alpha(\alpha+1)\,.\,\beta(\beta+1)}{\gamma\,.\,(\gamma+1)\,.\,1\,.\,2}x^2 + \ldots;$$

then it is well known that (Crelle's *Journal*, Bd. XV., p. 52)

$$F\{\alpha, \beta, \gamma, x\} = (1-x)^{-\beta}\,F\left\{\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right\},$$

whence, putting $\alpha = n-m+1$, $\beta = 1-m$, $\gamma = n+1$, $x = a^2$, we obtain the required result.]

6463. (By H. L. Orchard, M.A.)—Find the locus of the intersection of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of a chord through the point $(\alpha, 0)$.

Solution by R. KNOWLES, L.C.P.; R. E. RILEY, B.A.; *and others.*

The normals at the ends of the chord $l\frac{x}{a}+m\frac{y}{b}-1=0$ will meet in the point $x=-\frac{a^2-b^2}{a}\frac{l(m^2-1)}{l^2+m^2}$, $y=+\frac{a^2-b^2}{b}\frac{m(l^2-1)}{l^2+m^2}$.

The results of eliminating m, l between these two equations, are

$$\frac{l(l^2+1)^2}{(l^2-1)^2}\cdot\frac{b^2y^2}{(a^2-b)^2}=\left(l+\frac{ax}{a^2-b^2}\right)\left(1-\frac{alx}{a^2-b^2}\right),$$

$$\frac{m(m^2+1)^2}{(m^2-1)^2}\cdot\frac{a^2x^2}{(a^2-b^2)^2}=\left(m-\frac{by}{a^2-b^2}\right)\left(1+\frac{bmy}{a^2-b^2}\right).$$

If the chord pass through $(\alpha, 0)$, we have $l=\frac{a}{\alpha}$, and the locus is a conic.

Similarly, if it pass through $(0, \beta)$, we have $m=\frac{b}{\beta}$.

6433. (By J. W. MCKENZIE, B.A.)—Find, if possible, a value of n (other than $n=24$), so that the sum of the squares of the first n natural numbers may be a square.

Solution by G. HEPPEL, M.A.; G. EASTWOOD, M.A.; *and others.*

The sum of the squares of the first n natural numbers is $\frac{1}{6}n(n+1)(2n+1)$. Since n, $n+1$, $2n+1$ are all prime to one another; in order that the expression may be a square, each of the three must contain a square factor; and 6, or its factors 2 and 3, must be distributed among them as multipliers.

If $n=m^2$, then $n+1$ or m^2+1 cannot be of the form k^2 or $3k^2$ or $6k^2$, and the only possible case is

$$n=m^2,\ m+1=2k^2,\ 2m^2+1=3l^2 \quad \text{.....(A).}$$

If $n=2m^2$, $4m^2+1$ cannot be either l^2 or $3l^2$, and there is no possible case.

If $n=3m^2$, $3m^2+1$ cannot be either k^2 or $3k^2$, and there is no possible case.

If $n=6m^2$, then let $6m^2+1=k^2$, $12m^2+1=l^2$.....(B).

Taking case (B) first, the values of m, such that $6m^2+1$ is a square, are

0, 2, 20, 198, 1960, 19402, 192060, 1901198, &c.;

the law of formation being $u_n=10u_{n-1}-u_{n-2}$. The values of m, such that $12m^2+1$ is a square, are

0, 2, 28, 390, 5432, 75658, 1068780, &c.;

the law of formation being $u_n=14u_{n-1}-u_{n-2}$. The number 2 occurs in both these series, and in that case $n=6m^2=24$. But no other, less than a million, occurs in both, and therefore n cannot be less than six times the square of a million.

Taking case (A), we must first have $m^2+1=2k^2$; the values of m being

1, 7, 41, 239, 1393, 8119, 47321, 275807, and 1607521;

the law of formation being $u_n = 6u_{n-1} - u_{n-2}$; and, secondly, we must have the relation $2m^2 + 1 = 3l^2$.

$m = 1$ satisfies both conditions, and consequently gives a solution. The values of m that satisfy $2m^2 + 1 = 3l^2$, may, by solving the equations $2(m+l) = k(l+1)$, $k(m-l) = l-1$, be expressed generally thus

$$l = \frac{2+k^2}{2+4k-k^2}, \quad m = \frac{k^2+2k-2}{2+4k-k^2};$$

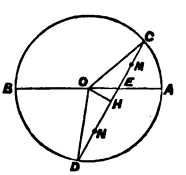
and, if $k = 4$, we have $m = 11$, $l = 9$. There is almost certainly no other solution in whole numbers. At any rate, if the numbers in the series last given be tested, they do not satisfy the relation $2m^2 + 1 = 3l^2$. Hence there is no solution under $n = (1{,}000{,}000)^2$.

[It is most probable that there is no solution other than $n = 24$.]

5708. (By Professor Seitz, M.A.)—Within a given circle a line is drawn at random in position and length; show that (1) the chance that the line intersects a given diameter is π^{-1}; and (2) the average length of the line is $\frac{3}{16}\pi r$.

Solution by the Proposer.

Let ABCD be the given circle, O its centre, AB the given diameter, and MN the random line. Produce MN, forming the chord CD, and draw OH perpendicular to CD. Let $OA = r$, $HM = x$, $MN = y$, $\angle COH = \theta$, and $\angle AEC = \phi$.



1. We have

$$OH = r\cos\theta, \quad CE = r(\sin\theta - \cos\theta\cot\phi),$$

and $$DE = r(\sin\theta + \cos\theta\cot\phi).$$

The number of ways the line MN can be drawn in the chord CD, so as to intersect AB, is $2CE \cdot DE = 2r^2(1-\cos^2\theta\operatorname{cosec}^2\phi)$, and the whole number of ways it can be drawn in the chord is $CD^2 = 4r^2\sin^2\theta$. The limits of θ are 0 and $\frac{1}{2}\pi$, and, in order that CD shall intersect AB, the limits of ϕ must be $\frac{1}{2}\pi - \theta$ and $\frac{1}{2}\pi + \theta$, and doubled. Hence the chance that MN intersects AB is

$$p = 2\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta} 2r^2(1-\cos^2\theta\operatorname{cosec}^2\phi)\, r\sin\theta\, d\theta\, d\phi + \int_0^{\frac{1}{2}\pi}\int_0^{2\pi} 4r^3\sin^3\theta\, d\theta\, d\phi.$$

$$= \frac{3}{4\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta}(1-\cos^2\theta\operatorname{cosec}^2\phi)\sin\theta\, d\theta\, d\phi$$

$$= \frac{3}{2\pi}\int_0^{\frac{1}{2}\pi}(\theta - \sin\theta\cos\theta)\sin\theta\, d\theta = \frac{1}{\pi}.$$

2. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and 2π; of x, $-r\sin\theta$ and $r\sin\theta$; and of y, 0 and $x + r\sin\theta = y_1$, and doubled; hence the average

length of MN is

$$\Delta = 2\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_{-r\sin\theta}^{r\sin\theta}\int_0^{y_1} yr\sin\theta\, d\theta\, d\phi\, dx\, dy \div \int_0^{\frac{1}{2}\pi}\int_0^{2\pi} 4r^3\sin^3\theta\, d\theta\, d\phi$$

$$= \frac{r}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\sin^4\theta\, d\theta\, d\phi = r\int_0^{\frac{1}{2}\pi}\sin^4\theta\, d\theta = \tfrac{3}{16}\pi r.$$

6154. (By Professor SYLVESTER, F.R.S.)—If the roots of a Quantic are all real, prove that the roots of its Hessian are all imaginary.

I. *Solution by the* PROPOSER.

Let $(a, b, c \ldots \chi x, y)^n$ be a Quantic, all whose roots are real; then, by NEWTON'S Rule, or by common sense, we know that $b^2 - ac$ must be positive; and obviously, if, when we write for y, $\lambda x + \mu y$, a, b, c become a', b', c' respectively, $b'^2 - a'c'$ also must be positive for all real values of λ, μ.

But, on making this substitution, and then replacing λ, μ by y and $-x$, $b'^2 - a'c'$ becomes the negative Hessian of the Quantic.

The Hessian, then, is always negative, and never changes its sign for any real value of $\frac{x}{y}$; consequently, its roots are all imaginary.

COR. 1.—Since the roots of every derivative of the Quantic, obtained by differentiating it any number of times in respect to x, and any number of times in respect to y, must also be all real, it follows that the roots of the Hessians of all these derivatives must be all imaginary. The number of these, including the given Quantic, but stopping short at the derivatives of the second degree in x, y, because linear functions have no Hessian, will be $1 + 2 + \ldots + (n-1)$ or $\frac{1}{2}(n-1)^2$. We thus obtain that number of essentially negative quantities, of which $(n-1)$ are the constants $c - b^2$, $(bd - c^2)$, ...; when these are left out, $\frac{1}{2}(n-2)(n-1)$ functions remain, of degrees ranging from $2n-4$ down to 2, all of which will have exclusively imaginary roots, in consequence of a single function of degree n having its roots exclusively real.

COR. 2.—Since it is easy to show, for the Quantic $(a, b, c, d, e, f, g \ldots \chi x, y)^n$ that the quantities $ae - 4bd + 3c^2$, $ag - 6bf + 15ce - 10d^2 \ldots$ must be of the same sign as their last terms when the roots of the Quantic are all real, it follows, by the same reasoning as was applied to the Hessian, that if the roots of a Quantic are all real, the roots of each and every of its Quadri-covariants (*i.e.*, covariants of the second degree in the coefficients), and of the Quadri-covariants of its differential Derivatives, are all imaginary.

II. *Solution by* H. STABENOW, M.A.; W. J. C. SHARP, M.A.; *and others.*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ represent the n real roots of a quantic of the same degree in the variables. Its Hessian—expressed in terms of $\alpha_1, \alpha_2, \ldots$—will be (with sign changed)

$$\Sigma\,(a_1-a_2)^2\,(x-a_3)^2\,(x-a_4)^2\ldots(x-a_n)^2,$$

that is to say, it will consist of $\frac{1}{2}n\,(n-1)$ squares of the $2\,(n-2)^{\text{th}}$ degree in x.

These squares being all positive in our hypothesis, it is obvious that the Hessian cannot vanish for any real value of x, unless for that same value each of the squares reduce to 0 separately, which is impossible when, as supposed in the proposition, the quantities $a_1, a_2, \ldots$ are all unequal.

But the Hessian equated to 0 has $2\,(n-2)$ roots; these roots, then, must be all imaginary.

Should any number m of roots be equal, $a_1 = a_2 = \ldots = a_m$, then $\frac{1}{2}m\,(m-1)$ squares will evidently vanish and the remaining ones will contain as a common factor the binomial $(x-a_1)^{2\,(m-1)}$, so that in this case the number of imaginary roots of the Hessian will be reduced to $2\,(n-m-1)$.

[We get the same result, if we actually form the Hessian H of $(x-a)^m\,\phi\,(x, y)$, ϕ being a quantic of n dimensions in x, y.

For, denoting the first and second differential coefficients of ϕ by $\phi_1, \phi_2, \phi_{11}$, &c., we find

$$\mathrm{H} = \left\{ \begin{array}{l} (\phi_{11}\phi_{22}-\phi_{12}^2)\,(x-ay)^2 + 2m\,(\phi_1\phi_{22}-a\phi_{11}\cdot\phi_2+a\phi_1\phi_{12}-\phi_2\phi_{12})\,(x-ay) \\ \qquad + (m)\,(m-1)\,\phi\times(a^2\phi_{11}+2a\phi_{12}+\phi_{22})-m^2\,(a\phi_1+\phi_2)^2 \end{array} \right\} \times (x-ay)^{2(m-1)}$$

$$= \frac{n+m-1}{n\,(n-1)}\cdot(x-ay)^{2(m-1)}$$

$$\times\left[(n+m)\,(\phi_{11}\phi_{22}-\phi_{12}^2)\,(x-ay)^2-\frac{m}{n-1}\left\{x\,(a\phi_{11}+\phi_{12})+y\,(a\phi_{12}+\phi_{22})\right\}^2\right],$$

in virtue of the known properties of homogeneous functions.

Since $\phi_{11}\phi_{22}-\phi_{12}^2$ is the Hessian of $\phi\,(x, y)$, and, as shown before, negative for any real value of x, the roots of the quantic within the brackets [ ] are necessarily all imaginary.]

6503. (By Sir James Cockle, F.R.S.)—Transform the equation (1) of Question 6404 into

$$\mathrm{Y}''+\left(\varpi\tan t-\frac{i\lambda}{\varpi}\right)\mathrm{Y}'+\tfrac{1}{4}\left\{1-\left(\varpi+\frac{\lambda}{\varpi}\right)^2-4\mathrm{N}\right\}\mathrm{Y}=0\ldots\ldots\ldots(\text{i.}),$$

where $i=(-1)^{\frac{1}{2}}$; show that (i.) admits of a continuous transformation whereby ϖ is changed into $\varpi-2$, $\varpi-4$, and so on; and notice the case of $\varpi=0$.

Solution by the Proposer.

1. Put $\mathrm{A}+\mathrm{E}=\mathrm{S}$, $\mathrm{A}-\mathrm{E}=\mathrm{D}$, $\tan x=\cos\theta+i\sin\theta$, and change the independent variable from x to θ; (1) becomes $y''+2qy'+ry=0$ .....(2), where $2q=(\mathrm{D}-1)\tan\theta+i\mathrm{S}$, $4r=(\mathrm{D}^2-\mathrm{\AE}^2)\,(\sec\theta)^2$; and if we write (i.) in the form

$$\mathrm{Y}''+2\mathrm{QY}'+\mathrm{RY}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

then (see *Reprint*, IX., 105—112) $e^{\int \mathrm{Q}d\theta}\,\mathrm{Y}=e^{\int q d\theta}y$, provided that

$$\mathrm{R}-\mathrm{Q}^2-\mathrm{Q}'=r-q^2-q';$$

viz., if $1-4N-2\lambda+2i\lambda\tan\theta-(\varpi^2+2\varpi)(\sec\theta)^2$
$= S^2+(D-1)^2-2iS(D-1)\tan\theta+(1-\text{Æ}^2)(\sec\theta)^2$
identically. This gives $(\varpi+1)^2=\text{Æ}^2$, $\lambda=-S(D-1)$,
$1-4N=(S-D+1)^2=(2E+1)^2$, or $N=-E(E+1)$.
If $L=A(1-A)$, then $\lambda=L-N$.

2. Differentiating (ii.), and putting $Y'=U$, we get
$$U''+2QU'+\{R+\varpi(\sec\theta)^2\}U=0.$$
Assume $U''_2+2Q_2U'_2+R^2U_2=0$, then will $e^{\int Q_2 d\theta}U_2=e^{\int Q d\theta}U$, provided that $R_2-Q_2^2-Q'_2=R-Q^2-Q'+\varpi(\sec\theta)^2$. This is satisfied if Q_2, R_2 be derived from Q, R by changing ϖ into $\varpi-2$, or into $-\varpi$.

3. When $\varpi=0$, then $\text{Æ}^2=1$, and (2) is soluble if $D^2=1$, or again, if $S=0$. For, if $S=0$, then (2) transforms into $4u''+(D-1)^2u=0$. The independent variable t of the Question is herein replaced by θ.

5493. (By Professor SYLVESTER, F.R.S.)—If χ_q represent in general the number of linearly independent covariants of the degree q in the variables, and of a given order j in the constants belonging to a binary quantic of the degree i, prove that
$$\chi_0+2\chi_1+3\chi_2+4\chi_3+5\chi_4+\ldots\ldots=\frac{\Pi(i+j)}{\Pi i\,.\,\Pi j}.$$

[Here χ_0 represents the number of invariants, and the series stops spontaneously after the $(ij+1)^{\text{th}}$ term by all the subsequent χ's vanishing. Of course, when ij is even all the odd-indiced χ's, and when ij is odd, all the even-indiced χ's, vanish. The theorem as stated comprehends both of these cases in one.]

Solution by W. J. CURRAN SHARP, M.A.

The number of covariants of order j and weight of source $p=\frac{1}{2}(ij-q)$ is (SALMON's *Higher Algebra*, p. 272) the difference of the coefficients of x^p and x^{p-1} in the expansion of
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x)(1-x^2)\ldots(1-x^j)}\equiv 1+A_1x+A_2x^2+A_3x^3+\&c.,$$
where A_1, A_2, &c. are all positive; and χ_p = the coefficient of $x^{\frac{1}{2}(ij-p)}$ in
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x^2)(1-x^3)\ldots(1-x^j)};$$
therefore $\chi_0+2\chi_1+3\chi_2+\&c.$ = the coefficient of $x^{\frac{1}{2}(ij)}$ in the product of this expression, and $1+2x^{\frac{1}{2}}+3x+\&c.$; $\left(i.e.\ \frac{1}{(1-x^{\frac{1}{2}})}\right)$; *i.e.*, the coefficient of $x^{\frac{1}{2}(ij)}$ in
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x)(1-x^2)\ldots(1-x^j)}\times\frac{(1+x^{\frac{1}{2}})^2}{1-x}=1+A_1+A_2+\ldots\ \&c.=\frac{\Pi(i+j)}{\Pi i\,.\,\Pi j},$$
since $A_r=A_{ij-r}$.

[A *Note* on this Question by the PROPOSER may be seen on p. 87 of Vol. XXXI. of the *Reprint*.]

5968. (By the late Prof. BENJAMIN PEIRCE, F.R.S.)—If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction.

Solution by Professor ASAPH HALL, M.A.

At p. 91 of Vol. XXXIII. of the *Reprints* from the *Educational Times*, I find the above Question by Professor PEIRCE answered by Mr. MONRO. As I understand the Question, it seems to me correct. Suppose any number of particles attracted toward a centre according to the given law, and that the particles do not act on each other. Then the equations of motion for any particles are $\frac{d^2x}{dt^2}+kx=0$, $\frac{d^2y}{dt^2}+ky=0$, $\frac{d^2z}{dt^2}+kz=0$.

The solution of these equations shows that each particle moves in an ellipse which has its centre at the centre of this attracting force. For the relative motion of two particles, this equation will be

$$\frac{d^2(x-x_1)}{dt^2}+k(x-x_1)=0,\quad \frac{d^2(y-y_1)}{dt^2}+k(y-y_1)=0,$$
$$\frac{d^2(z-z_1)}{dt^2}+k(z-z_1)=0;$$

and since these equations have the same form as the first, we see that each particle will describe an ellipse around every other particle, the centre of the ellipse being at the particle. Now, let μ be the mutual attraction of two particles, r their distance apart, and let x, y, z be the coordinates of one particle with respect to another. The equations for the relative motion are

$$\frac{d^2x}{dt^2}+kx+\frac{\mu x}{r}=0,\quad \frac{d^2y}{dt^2}+ky+\frac{\mu y}{r}=0,\quad \frac{d^2z}{dt^2}+kz+\frac{\mu z}{r}=0.$$

Multiply the second equation by z, and the third by y; we get, after integration, and two more similar combinations,

$$z.\frac{dy}{dt}-y.\frac{dz}{dt}=c,\quad x.\frac{dz}{dt}-z.\frac{dx}{dt}=c',\quad y.\frac{dx}{dt}-x.\frac{dy}{dt}=c''.$$

Hence we have $cx+c'y+c''z=0$; or the motion is in a plane, and this plane passes through the particle. [Mr. MONRO remarks that, according to Prof. HALL, $\frac{d^2x}{dt^2}:x=\frac{d^2x_1}{dt^2}:x_1$; that is, the *acceleration* towards the centre is independent of mass. Mr. MONRO had shown that the problem was correct when "force" was understood as acceleration.]

6248. (By the EDITOR.)—If two points P, Q be taken at random on the area of a vertical circle; show that the probability that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{2}$.

Solution by G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others.*

Let (r, θ) be the polar coordinates of P, and x the radius of the circle whose highest point is P, and touching the other O; then we have

$$(a-x)^2 = x^2 + r^2 - 2xr\sin\theta,$$

therefore $$2x = \frac{a^2-r^2}{a-r\sin\theta},$$

hence the required chance is

$$\iint \frac{\frac{x^2}{a^2}\, r\,d\theta\,dr}{\pi a^2} = \frac{1}{4\pi a^4}\iint\left[\frac{a^2-r^2}{a^2-r\sin\theta}\right]^2 r\,d\theta\,dr$$

$$= \frac{1}{4\pi a^4}\cdot 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^a \left[\frac{a^2-r^2}{a-r\sin\theta}\right]^2 r\,d\theta\,dr = \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^1 \left[\frac{1-r^2}{1-r\sin\theta}\right]^2 r\,d\theta\,dr$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-r^2)^2\, r\,dr\,d\theta\left[\frac{1}{(1-r\sin\theta)^2}+\frac{1}{(1+r\sin\theta)^2}\right]$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-r^2)^2 r\,dr\,d\theta\,\frac{1+r^2\sin^2\theta}{(1-r^2\sin^2\theta)^2} = \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-x)^2\,dx\,d\theta\,\frac{1+x\sin^2\theta}{(1-x\sin^2\theta)}$$

$$= \frac{1}{2\pi}\int_0^1\int_0^\infty (1-x)^2\,dx\,dy\,\frac{1+y^2+x}{(1+y^2-x)^3},\quad [y=\cot\theta]$$

$$= \frac{1}{2\pi}\int_0^1\int_0^\infty dx\,(1-x)^{\frac{1}{2}}\,dv\,\frac{1+x+(1-x)\,v^2}{(1+v^2)^3},\quad [y=(1-x)^{\frac{1}{2}}v]$$

$$= \frac{1}{2\pi}\int_0^1\int_0^\infty dx\,(1-x^4)^{\frac{1}{2}}\,dv\,\frac{(1+v^2)+(1-v^2)\,x}{(1+v^2)^3}$$

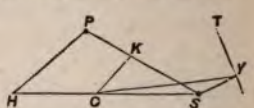
$$= \frac{1}{2\pi}\int_0^1 dx\,(1-x)^{\frac{1}{2}}\int_0^\infty\left[\tan^{-1}v+\frac{vx}{1+v^2}\right] = \frac{1}{2\pi}\int_0^1 dx\,(1-x)^{\frac{1}{2}}\,(\tfrac{1}{2}\pi)$$

$$= \frac{1}{2\pi}\int^1 dx\,(x)^{\frac{1}{2}}\,(\tfrac{1}{2}\pi) = \frac{1}{2\pi}\cdot\frac{\pi}{2}\cdot\frac{2}{3} = \frac{1}{6}.$$

6480. (By Professor GENESE, M.A.)—Given a tangent, a point, and a focus of a conic; the locus of the centre is a conic.

Solution by CHARLOTTE A. SCOTT; C. BICKERDIKE, B.A.; *and others.*

Let P be the given point, S the focus, Y the foot of the perpendicular from S on the given tangent YT, H one position of the other focus, and C the corresponding centre. Then, (1) if S, H be on same side of YT, P lies on an ellipse with foci S, H; and (2) if S, H be on opposite sides of YT, P lies on an hyperbola with foci S, H; and, in each case, CY is the major semi-axis.



Let K be the middle point of PS; then $CK = \frac{1}{2}HP$; therefore (1) $CK + KS = \frac{1}{2}(HP + PS) = CY$, $\therefore CY - CK = KS$; thus the locus of C is an hyperbola with foci K, Y, and transverse axis = KS. Again, (2) $CK \sim KS = \frac{1}{2}(HP \sim SP) = CY$, $\therefore CK \mp CY = KS$ (the upper or lower signs being taken according as P and S are on the same or opposite sides of TY). Finally, the locus of C is a conic, with foci at K and Q, and major axis $= \frac{1}{2}SP$, and is a hyperbola or ellipse, according as S and P are on the same or opposite sides of TY.

6489. (By T. P. Kirkman, M.A., F.R.S.)—On a square chequer of n^2 small squares, stand $n^2 - 1$ counters marked 1, 2, 3 ... $(n^2 - 1)$, in any order S, covering all the squares except the last, at the right-hand corner; this last is occupied by a King, visible or invisible. The King, on whatever square he happens to stand, may exchange places only with any counter on a compartment collateral with his. The problem for solution is, to move the King about the board so that, when he has returned to his own final square, the order S shall be reduced to the natural order, the counters marked a, b, c... standing on the a^{th}, b^{th}, c^{th} ... squares. Required a rule whereby, on inspection of the order S, the solution can be proved possible or impossible.

[This problem is a generalized form of the famous American puzzle relating to 15 counters on 16 squares.]

Solution by the Proposer.

The given order S is a substitution made with n^2 elements, of which the final one, the King, is undisturbed. The only substitution which, operating on S, will restore the natural order, is S^{-1}, made with the same n^2 elements. If this restoration results from the movements of the King, their product as substitutions is equal to S^{-1}.

The King can make no excursion from home and back again, but by an even number of simple transpositions; for every step from home, whether vertical or horizontal, must be followed by an equal and parallel step in the opposite direction. It follows that S^{-1} is a positive substitution, *i.e.*, made by an even number of simple transpositions. Consequently S is positive, when the solution is possible.

If, then, the given S is negative, that is, if it has an odd number of circular factors, each of an even number of elements, a thing evident to inspection, the solution is impossible.

When S is positive, whatever be n^2, the King can dance round and head up every counter to its place, to rest there. When S is negative he can do this only for all the counters but the two last, which remain transposed.

An example on the chequer of nine squares is the following:—

Let S = 123586749, which is positive. King Nine has to take fourteen steps, which, written as substitutions, are

123956784	123456798	123496785	123956784
123456798	123956784	123956784	123496785
123496785	123459786	123456798	123459786
123456987		123456987	

Operating on the first by the second, on their product by the third, &c., the product of the fourteen is found to be $S^{-1} = 123846759$.

6499. (By J. L. McKenzie, B.A.)—If x_n and y_n be the n^{th} positive integral solution of $x^2 - Ny^2 = 1$, prove that $x_{n+p} = 2x_p x_n - x_{n-p}$, and $y_{n+p} = 2x\ y_n - y_{n-p}$.

Solution by Prof. Arnold Droz; L. W. Jones, B.A.; *and others.*

Soient x_1, y_1 les deux plus petites racines de l'équation $x^2 - Ny^2 = 1$, les racines x_n et y_n seront données par l'équation $(x_n + y_n N^{\frac{1}{2}}) = (x_1 + y_1 N^{\frac{1}{2}})^n$;

donc $x_{n+p} + y_{n+p} N^{\frac{1}{2}} = (x_1 + y_1 N^{\frac{1}{2}})^{n+p} = (x_1 + y_1 N^{\frac{1}{2}})^n (x_1 + y_1 N^{\frac{1}{2}})^p$

$= (x_n + y_n N^{\frac{1}{2}})(x_p + y_p N^{\frac{1}{2}}) = (x_n x_p + y_n y_p N) + (x_n y_p + x_p y_n) N^{\frac{1}{2}}$;

$$x_{n+p} = x_n x_p + y_n y_p N,\quad y_{n+p} = x_n y_p + x_p y_n \ldots\ldots\ldots\ldots (1).$$

On a de même

$$x_{n-p} + y_{n-p} N^{\frac{1}{2}} = (x_1 + y_1 N^{\frac{1}{2}})^{n-p} = \frac{(x_1 + y_1 N^{\frac{1}{2}})^n}{(x_1 + y_1 N^{\frac{1}{2}})^p} = \frac{(x_1 + y_1 N^{\frac{1}{2}})^n (x_1 - y_1 N^{\frac{1}{2}})^p}{(x_1 - N y_1^2)^p};$$

mais $x_1^2 - N y_1^2 = 1$, et $(x_1 - y_1 N^{\frac{1}{2}})^p = x_p - y_p N^{\frac{1}{2}}$, on aura

$$x_{n-p} + y_{n-p} N^{\frac{1}{2}} = (x_n + y_n N^{\frac{1}{2}})(x_p - y_p N^{\frac{1}{2}})$$
$$= (x_n x_p - y_n y_p N) + (x_p y_n - x_n y_p) N^{\frac{1}{2}};$$

$$x_{n-p} = x_n x_p - y_n y_p N,\quad y_{n-p} = x_p y_n - x_n y_p \ldots\ldots\ldots\ldots (2).$$

En comparant les équations (1) et (2), on obtient

$$x_{n+p} = 2x_n x_p - x_{n-p},\quad y_{n+p} = 2x_p y_n - y_{n-p}.$$

6219. (By Professor Townsend, F.R.S.)—A thin uniform spherical shell being supposed to hold in free equilibrium a material particle, situated at its centre of form, by the attraction of its mass for a single power, direct or inverse, of the distance; determine the groups of powers for which the equilibrium of the particle is respectively stable and unstable.

Solution by the Proposer.

Denoting by a the radius of the shell, by m its mass, by n the index of the law of force, and by R the internal attraction of the mass at the distance x from its centre; then, since, as may be easily shown,

$$R = (n+2) m a^n \left[\frac{n_0}{1.3} \left(\frac{x}{a}\right)^1 + \frac{n_2}{3.5} \left(\frac{x}{a}\right)^3 + \frac{n_4}{5.7} \left(\frac{x}{a}\right)^5 + \&c. \right],$$

where n_0, n_1, n_2, &c. are the several binomial coefficients for the index n, therefore, for a point so near the centre that all powers above the first of the ratio $x : a$ may be neglected, we have $R = \frac{1}{2}(n+2)ma^{n-1}x$; which being positive or negative with $n+2$, therefore, &c.

When $n+2=0$, that is, for the ordinary law of the inverse square of the distance, then, as is well known, $R = 0$ for all values of x internal to the shell.

6281. (By Prof. WOLSTENHOLME, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a circle whose centre lies on the given asymptote.

Solution by the Rev. F. D. THOMSON, M.A.; W. GALLATLY, B.A.; *and others.*

Let the given asymptote be the axis of y, and (a_1b_1) (a_2b_2) the two given foci, (XY) a third focus, then the equation of the cubic is

$$lr_1 + mr_2 = (l+m)r_3,$$

where r_1, r_2, r_3 are the distances from the three foci, and

$$r_1 = \{(x-a_1)^2+(y-b_1)^2\}^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{2}}\left(1 - \frac{2a_1x+2b_1y-a_1^2+b_1^2}{2(x^2+y^2)} + \dots\right),$$

and similarly for the others. Hence the equation of the real asymptote is

$$l(2a_1x+2b_1y-a_1^2-b_1^2)+m(2a_2x+2b_2y-a_2^2-b_2^2)$$
$$= (l+m)(2Xx+2Yy-X^2-Y^2);$$

and since this coincides with the axis of y,

$$lb_1+mb_2 = (l+m)Y,\ l(a_1^2+b_1^2)+m(a_2^2+b_2^2) = (l+m)(X^2+Y^2);$$

and the locus is the *circle*

$$(X^2+Y^2-a_1^2-b_1^2)(b_2-Y) = (X^2+Y^2-a_2^2-b_2^2)(b_1-Y),$$

that is, the circle which passes through the two given foci, and has its centre on the given asymptote.

6483. (By R. TUCKER, M.A.)—From P, any point on the line $x=2a$, tangents PT, PT′ are drawn to a parabola; the normals at T, T′ (as is known) intersect on the curve (in Q). Prove that TT′ passes through a fixed point on the axis, on which axis also lies the centroid of Q, T, T′. the circle about PTQT′ passes through the vertex, and its envelope is a cubic curve; and the orthocentre of QTT′ lies on the diameter through P and on a parabola.

Solution by D. EDWARDES; J. O'REGAN; *and others.*

Let m_1, m_2 be the parameters of the points T, T′. Let H be the orthocentre in question, O the "equidistant angle point," G the centroid. Since

P lies on the line $x = 2a$, we have $m_1 m_2 = 2$. The equation of TT′, viz. $2x - (m_1 + m_2)y + 2am_1 m_2 = 0$, shows that TT′ meets the axis in the fixed point $-2a$. Also, the coordinates of Q are easily found to be $\frac{1}{2}(a\lambda^2)$, $-a\lambda$, where $\lambda = 2(m_1 + m_2)$. Let x_P, y_P denote the coordinates of any point P. Then $3y_G = 2a(m_1 + m_2) - a\lambda = 0$; therefore G lies on the axis. Again, the equation of circle about PTQT′ is

$$(x - x_Q)(x - x_P) + (y - y_Q)(y - y_P) = 0, \text{ or } ax\lambda^2 - 2ay\lambda + 4(2ax - x^2 - y^2) = 0;$$

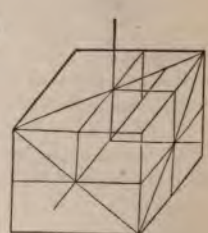
its envelope is therefore the cubic $ay^2 + 4x(x^2 + y^2 - 2ax) = 0$. Let A be the vertex, then the equation of AQ is $\lambda y + 4x = 0$, and of AP is $y = \frac{1}{2}(m_1 + m_2)x = \frac{1}{4}\lambda x$; therefore PAQ is a right angle, and the circle passes through A. Now $x_G = \frac{1}{8}a(\lambda^2 - 8)$, $y_G = 0$, $x_0 = a(1 + \frac{1}{8}\lambda^2)$, $y_0 = -\frac{1}{4}a\lambda$; therefore, by a known geometrical theorem,

$$x_H = \tfrac{1}{4}a\lambda^2 - 6a, \quad y_H = \tfrac{1}{2}a\lambda = y_P.$$

Hence H lies on the diameter through P, and on the parabola $y^2 = a(x + 6a)$.

6414. (By Elizabeth Blackwood.)—If x, y, z be each taken at random between 1 and -1, find the chances (1) that $x + y$, $y + z$, $z + x$ are all three positive, (2) that $x + y - xy$, $y + z - yz$, $z + x - zx$ are all three positive.

I. *Solution by* G. F. Walker, M.A.; H. Fortey, M.A.; *and others.*



1. The chance will be the ratio of the volume of the cube

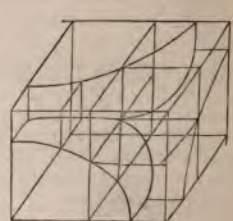
$$x = \pm 1, \; y = \pm 1, \; z = \pm 1,$$

to the positive side of the planes

$$y + z = 0, \; z + x = 0, \; x + y = 0,$$

to the whole volume; and therefore chance is

$$\frac{3 \cdot \frac{1}{8} + 1}{(2)^3} = \frac{1}{4}.$$



2. The chance will be the ratio of the volume cut off from the cube to the positive side of the hyperbolic cylinders

$$yz = y + z, \; zx = z + x, \; xy = x + y.$$

The finite intersections of these are along the planes $y = z, \; z = x, \; x = y$;

and the volume cut off is

$$1 + 3\int_{-1}^{0}\left(1 - \frac{x}{x-1}\right)^2 dx = 1 + 3\int_{-1}^{0} -\frac{1}{x-1}$$

$$= 1 + 3(1 - \tfrac{1}{2});$$

and chance is

$$\frac{1 + 3(1 - \frac{1}{2})}{(2)^3} = \frac{5}{16}.$$

II. *Solution by* W. B. Grove, B.A.; Colonel J. H. Fry; *and others.*

The chance (1) required is $\dfrac{AQ\iiint dx\,dy\,ds}{A\iiint dx\,dy\,dz}$,

where A denotes the statement $x_{1'.2}\ y_{1'.2}\ z_{1'.2}$ (Table I.),

and $$Q = p(x+y)\,p(y+z)\,p(z+x) = y_3 z_{3.4}.$$

Table I.

$x_1 = 1$	$y_1 = 1$	$z_1 = 1$
$x_2 = -1$	$y_2 = -1$	$z_2 = -1$
	$y_3 = -x$	$z_3 = -y$
	$y_4 = x$	$z_4 = -x$

The limits of integration are readily found by Mr. McColl's method, and the result is

$$AQ = x_{1'.0}\,y_{4'.3}\,z_{1'.3} + x_{0'.2}\,y_{1'.4}\,z_{1'.4} + x_{1'.0}\,y_{1'.4}\,z_{1'.4},$$

so that chance (1) $= \frac{1}{8}(1+\frac{1}{3}+\frac{2}{3}) = \frac{1}{4}$.

Table II.

$x_1 = 1$	$y_1 = 1$	$z_1 = 1$
$x_2 = -1$	$y_2 = -1$	$z_2 = -1$
$x_3 = \frac{1}{2}$	$y_3 = -\dfrac{x}{1-x}$	$z_3 = -\dfrac{y}{1-y}$
	$y_4 = \frac{1}{2}$	
	$y_5 = x$	$z_4 = -\dfrac{x}{1-x}$

For chance (2), A will have the same as before; but (referring to Table II.)

$$Q = p(x+y-xy)\,p(y+z-yz) \times p(z+x-zx) = y_3 z_{3.4}.$$

Hence $AQ = x_{1'.2}\,y_{1'.2.3}\,z_{1'.2.3.4}$, that is,

$$AQ = x_{1'.3}\,y_{1'.4}\,z_{1'.2} + x_{1'.3}\,y_{4'.2}\,z_{1'.3} + x_{3'.0}\,y_{5'.3}\,z_{1'.3} + x\ \ y_{1'.3}\,z_{1'.4} + x_{3'.0}\,y_{1'.5}\,z_{1'.4};$$

so that chance (2) $= \frac{1}{8}\{\frac{1}{2} + \log 2 + (1-\log 2) + \frac{1}{3} + \frac{1}{2}\} = \frac{5}{16}$.

6484. (By W. H. Besant, M.A.)—Having given two circles of radii R and r, and that the distance between their centres is $(R^2-2Rr)^{\frac{1}{2}}$, prove that an infinite number of triangles can be drawn which shall be inscribed in one circle and circumscribed about the other.

I. *Solution by* D. Edwardes; Prof. Nash, M.A.; *and others.*

Let the equations of the circles be

$$S \equiv x^2+y^2-r^2 = 0, \quad S' \equiv (x-\delta)^2+y^2-R^2 = 0.$$

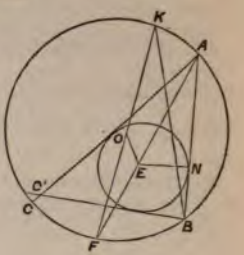
Calculating the invariants, we have

$$\Theta = \frac{\delta^2-R^2}{r^4} - \frac{2}{r^2}, \quad \Delta = -\frac{1}{r^4}, \quad \Theta' = \frac{\delta^2}{r^2} - 1 - \frac{2R^2}{r^2}.$$

The condition $\Theta^2 = 4\Delta\Theta'$ leads to

$$(\delta^2-2r^2-R^2) + 4(\delta^2r^2-r^4-2r^2R^2) = 0, \quad \text{or} \quad \delta^2 = R^2 \pm 2rR.$$

II. *Solution by the* PROPOSER.



Draw any chord AB touching the inner circle, and draw the tangents AC'C, BC'. The rest of the construction being obvious from the figure, we have

$$2Rr = R^2 - OE^2 = AE \,.\, EF,$$

therefore $EN : EA = EF : 2R$.

Now the angle $FKB = EAN$;

$\therefore$ $EN : EA = FB : FK$, and $EF = FB$.

Hence the angle $C'BF = EBF - \frac{1}{2}B$

$= FEB - \frac{1}{2}B = EAB = AF = CBE$,

therefore C and C′ coincide.

6400. (By J. HAMMOND, M.A.)—Prove that the surface

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is one of revolution, and find its axis and the equation of the generating curve (referred to its asymptotes as axes).

Solution by Rev. T. W. OPENSHAW, M.A.; J. O. JELLY, B.A.; *and others.*

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)[x^2 + \ldots - yz - \ldots]$$
$$= \tfrac{1}{2}(x+y+z)[3(x^2+y^2+z^2) - (x+y+z)^2].$$

Hence, changing to plane $x+y+z = 0$, as plane $X = 0$, we get

$$[X\sqrt{3}]\{3(X^2+Y^2+Z^2) - (X\sqrt{3})^2\} = 2a^3, \quad X(Y^2+Z^2) = \frac{2a^3}{3\sqrt{3}};$$

hence $x = y = z$ is the axis of rovolution, and the curve referred to its asymptotes is $xy^2 = \dfrac{2a^3}{3\sqrt{3}}$.

6326. (By the Rev. F. D. THOMSON, M.A.)—A conic cuts a cubic in 6 points; prove that the conics of 5-pointic contact at these points meet the cubic again in 6 points on a conic.

Solution by the PROPOSER; G. TORELLI; *and others.*

Let a, b, c, d, e, f be the 6 points, and let a', a'' be the 1st and 2nd tangentials of a. Then the *residual* of a'', a is the point where the conic of

5-pointic contact at a meets the cubic again (SALMON's *Higher Curves*, Art. 155).

But a', b', c', d', e', f' lie on one conic, for the conic $abcdef$ taken twice, and the conic through 5 points a', b', c', d' e' form a system of the 6th degree passing through $3 \times 6 - 1$ points, and therefore passing through one other fixed point on the cubic (SALMON, Art. 156). But the 6 tangents, aa', bb', &c. form another system of 6th degree through the same 17 points, and therefore f' is also on the conic. The same reasoning proves that a'', b'', c', d'', e'', f'' lie on a conic, and also that the residuals of a, a'', b, b'', &c. lie on another conic.

[If a, b, c be in a straight line, the same reasoning shows that a'', b'', c'' are in a straight line, and therefore the residuals of a, a'', b, b'', &c. are in a straight line, which furnishes a solution to Quest. 6210.]

6342. (By Prof. GENESE, M.A.)—Prove that the envelop of the directrix of a parabola (1) inscribed in, or described about, a triangle, is a conic; (2) touching two straight lines and passing through a point, or passing through two points and touching a straight line, is a quartic.

Solution by the PROPOSER.

The results are given incorrectly; the following is the method employed. With the usual notation, the equation to the directrix of a parabola represented by the general equation, is $2Gx + 2Fy = A + B$. That the directrix should pass through a given point P is therefore a line-condition. (WHITWORTH's *Trilinear Coordinates*, Chap. XXV.) That the curve is a parabola is another line-condition. With these conditions, (α) *one* conic only can be drawn to touch three straight lines; (β) *two* conics to touch two straight lines and pass through a given point; and (γ) *four* conics to touch one straight line and pass through two points, or to pass through three given points. Therefore, as regards the envelop, from any point P there can be drawn one, two, or four directrices of the systems (α), (β), (γ) respectively. The envelopes are therefore (α) a point, (β) a conic, or (γ) a curve of the fourth class.

6468. (By A. MARTIN, M.A.)—An auger hole is made through the centre of a sphere; show that the chance that the volume removed exceeds one-n^{th} of the sphere is $1-\left[1-\left(1-\frac{1}{n}\right)^{\frac{3}{2}}\right]^{\frac{1}{3}}$.

Solution by PROF. MATZ, M.A.; Prof. Evans, M.A.; *and others.*

Put a = the radius of the sphere; b = the radius of the auger-hole; and V = the volume removed. Then, we readily find $V = \frac{4}{3}\pi[a^3-(a^2-b^2)^{\frac{3}{2}}]$; therefore $b = a\left[1-\left(1-\frac{1}{n}\right)^{\frac{2}{3}}\right]^{\frac{1}{2}}$, and chance $= \frac{a-b}{a} =$ &c.

6362. (By E. B. Elliott, M.A.)—If m lines, straight or curved, are drawn across an enclosed portion of area; prove that, if n_1 be the number of simple internal intersections of these lines with one another or themselves, n_2 the number of triple points, n_3 of quadruple, &c., formed by them, the number of separate areas into which they divide the inclosure is

$$1+m+n_1+2n_2+3n_3+\ldots.$$

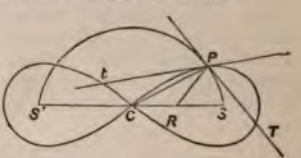
Solution by W. B. Grove, B.A.; E. Rutter; *and others.*

Consider first two lines which do not intersect, then divide any enclosed area into 3 parts. Now let them intersect; the number of parts becomes 4, therefore every point of simple intersection that is introduced increases the number of parts by 1. Similarly, it may be seen that every triple point introduced increases the number of parts by 2, and so on. Also m lines not intersecting divide the area into $(m+1)$ parts, hence the total number of parts is $1+m+n_1+2n_2+3n_3+\ldots.$

6457. By E. B. Elliott, M.A.)—P is one of the intersections of a lemniscate with the circle on the intercept SS′ between its foci as diameter. Prove that the two curves cut at an angle of 60°, and that the bisector of the angle SPS′ is also the bisector of the angle between the two tangents at P.

Solution by E. W. Symons, M.A.; D. Edwardes; *and others.*

Let the equation of the lemniscate be $r^2 = a^2\cos 2\theta$. At the point where it meets the semicircle in SS′, we must put $r = \frac{1}{2}a\sqrt{2}$, whence $\theta = 30°$. Now, $\angle CPZ = 90° - 2\angle SCY$; hence the angle between the tangents is $2\angle SCY$, that is, 60°.



Again, PT being the tangent to the circle at P, and PR the bisector of the angle between the tangents,

$$\angle SPR = 60° - SPT;\ \text{but}\ SPT = 90° - \angle SPC = 15°;$$

therefore $\angle SPR = 45°$, therefore PR bisects also the angle SPS′.

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4350. (Colonel Clarke, C.B., F.R.S.)—Three points are taken at random in a given triangle; prove that the probability (p) that they enclose a given point whose triangular coordinates are (a, b, c), is

$$p = \mathrm{F}(a, b, c) + \mathrm{F}(b, c, a) + \mathrm{F}(c, a, b),$$

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4706. (S. Tebay, B.A.)—A bag contains n tickets, marked with the numbers 1, 2, 3, ... n, representing prizes of the respective values a_1, a_2, a_3, ... a_n. A person draws m tickets, and selects the highest, the rest being replaced; a second person draws m tickets, and selects the highest, the rest being replaced; and so on for three or more persons. Find the expectations. 108

4753. (Prof. Crofton, F.R.S.)—If 1, 2, 3, 4 are four concyclic points on a Cassinian oval whose foci are F, F′, show that a bicircular oval can be drawn with 1, 2, 3, 4 as foci to pass through F, F′. Show also that the tangents at F, F′ are *double tangents;* each touching the bicircular oval in a second point, and each passing through the centre of the circle 1234... 52

4795 & 4198. (Prof. Crofton, F.R.S.)—Prove that—(1) the arc of a Cartesian oval, at any point p, is equally inclined to the straight line from p through any one point, and to the circular arc from p through the other two foci; and (2) any system of triconfocal Cartesian ovals intersect each other orthogonally. 40

4994. (Prof. Sylvester, F.R.S.) — The coordinates of a unicursal quartic x, y, z are given proportional to PQ′, P′Q, QQ′, where

$P = a + bt + ct^2$, $P' = a' + b't + c't^2$, $Q = \alpha + \beta t + \gamma t^2$, $Q' = \alpha' + \beta' t + \gamma' t^2$.

Show that two of the nodes are the intersections of xy with z, and that the coordinates of the third node are in the proportion of LM′ : L′M : MM′, where L, M, L′, M′ are the four first minors taken in a certain order of the rectangular matrix $\begin{vmatrix} a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}$ 109

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5121. (S. Tebay, B.A.)—If n letters be formed in groups of p letters in each, no q letters being repeated, but every q of the n letters exhibited; then, on erasing x letters, show, x being less than q, that the number of groups containing r of the x letters is

$$N_r = \frac{x!}{r!\,(x-r)!} \cdot \frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-r)!} \cdot \frac{(n-p)!}{(n-p-x+r)!}. \quad \text{..... } 70$$

5180. (Prof. Wolstenholme, M.A.)—A circular cubic whose four real foci lie on a circle is the locus of a point P, which moves so that AP . BP = CP . DP, where A, B, C, D are four fixed points on the focal circle, and there are for each curve two sets of such points. Given the foci of the curve, show how to obtain these points; or, given one set of the points, show how to obtain the other set and the foci. 54

5194. (A. W. Panton, M.A.)—If the coordinates (x, y, z) of a point on a unicursal cubic be proportional to $a_1\theta^3 + b_1\theta^2 + c_1\theta + d_1$, $a_2\theta^3 + b_2\theta^2 + c_2\theta + d_2$,

$a_3\theta^3 + b_3\theta^2 + c_3\theta + d_3$; show that the three values of θ at the points of inflexion are the roots of the cubic equation

$$\begin{vmatrix} 1 & -3\theta & 3\theta^2 & -\theta^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 73$$

5234. (S. Watson.)—Three points are taken at random within a given triangle; prove that the chance that they will all lie on one side of some one line that can be drawn through the centroid of the triangle is $\frac{25}{27} - \frac{20}{81}\log 2$. 34

5248. (Prof. Wolstenholme, M.A.) — If O, A, B be three points in order on a straight line, and a point P move so that OP is always a mean proportional between AP, BP; prove that (1) there exists also another system of three points A′, O′, B′, in order on the same straight line, such that O′P is a mean proportional between A′P, B′P; (2) O, O′ divide AB and A′B′ harmonically, and

$$OA + OB : OA' + OB' = OA \,.\, OB : OA' \,.\, OB' = (OA + OB)^2 : AB^2;$$

(3) the four points A, B, A′, B′ are the axial foci of the curve, which is a circular cubic, and the circle on OO′ as diameter is one of the circles with respect to which the curve is its own inverse passing through four impossible foci; and (4) the vector equations of the curve are, if $AP = r_1$, $BP = r_2$, $A'P = r_3$, $B'P = r_4$, and $4n = \dfrac{AB^2}{OA \,.\, OB}$,

$$2(n-1)^{\frac{1}{2}}r_2 + [2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_3 + [-2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_4 = 0,$$
$$2(n-1)^{\frac{1}{2}}r_1 + [2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_3 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_4 = 0,$$
$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]r_2 + 2(n+1)^{\frac{1}{2}}r_4 = 0,$$
$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_2 + 2(n+1)^{\frac{1}{2}}r_3 = 0.$$
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5309. (Prof. Wolstenholme, M.A.) — The two concentric squares ABCD, A′B′C′D′ have their sides parallel, and the side of the larger equal to the sum of side and diagonal of the smaller; prove that (1) an infinite number of sets of four points a, b, c, d can be taken on the sides of either, such that the lines joining them in order, ab, bc, cd, da, pass through the corners of the other [thus, a, b, c, d lie on AB, BC, CD, DA, and ab, bc, cd, da pass through B′, C′, D′, A′; also a_1, b_1, c_1, d_1 lie on A′B′, B′C′, C′D′, D′A′, and a_1b_1, b_1c_1, c_1d_1, d_1a_1 pass through D, A, B, C]; and (2) $abcd$ and $a_1b_1c_1d_1$ are each a mean proportional between the two..... 62

5344. (S. Tebay, B.A.)—n grocers have a_1, a_2, ... a_n lbs. of tea respectively ($a_1 < a_2 <$ &c.); all sell a certain number of lbs. at the same price, and afterwards the remainders at a different price, and all realize equal sums from the two sales. Show how this can be done, and deduce a simple particular solution. 67

5388 & 6471. (Prof. Crofton, F.R.S.) — A triangle Δ is divided into two portions by a line joining two points taken at random within it; prove that the mean value (1) of the triangular portion is $\frac{4}{9}\Delta$; (2) of the portion that contains the centroid of the triangle is $\frac{\Delta}{3^6}(470 + \frac{8}{3}\log 4)$; and (3) of the triangle formed by joining the centroid and the two points is $\frac{\Delta}{3^6}(43 + \frac{1}{3}\log 2)$. 25

5422. (Prof. Townsend, F.R.S.)—Four points A, B, C, D being supposed given or taken arbitrarily on a straight line L; construct, by elementary geometry, the two X and Y, on the line, the distance of each of which from one of the four D shall be the harmonic mean of its distances from the remaining three A, B, C..... 87

5433. (Prof. Pratt, M.A.) — Given $u = F(y)$, $y = F'\{z + xvf(y)\}$, $v = F_1 t$, $t = F_2\{z + xvf_1(t)\}$; expand u in a series of ascending, positive, integral powers of x (x not being a function of z). 77

5454. (Prof. Townsend, F.R.S.)—A solid ellipsoid of uniform density being supposed to revolve round its least axis of figure, and to carry with it a surrounding envelope of homogeneous incompressible fluid of different density, the entire mass attracting according to the ordinary law of the inverse square of the distance; required the several conditions requisite to the permanent assumption of the ellipsoidal form by the free surface of the fluid. 58

5533. (The Editor.)—Through two points taken at random anywhere inside a circle a chord is drawn, and a second chord through two other such random points; show that (1) the probability that these two chords will intersect inside the circle is $\frac{1}{3} + \frac{245}{72\pi^2}$; also (2), if through a third pair of such random points, a third chord be drawn, show that the respective probabilities that the three chords will intersect inside the circle in 0, 1, 2, 3 points are

$$p_0 = \frac{1}{3} - \frac{245}{36\pi^2} + \frac{23023}{576\pi^4}, \quad p_1 = \frac{2}{5} + \frac{155}{36\pi^2} - \frac{55055}{864\pi^4},$$

$$p_2 = \frac{1}{5} + \frac{115}{72\pi^2} + \frac{13013}{1728\pi^4}, \quad p_3 = \frac{1}{15} + \frac{65}{72\pi^2} + \frac{7007}{432\pi^4}. \text{ ..... } 56$$

5574. (R. Tucker, M.A.)—Three particles are projected simultaneously in the same vertical plane, with velocities v_1, v_2, v_3, at inclinations α_1, α_2, α_3 to the horizon; show that, when their directions are parallel,

$$\frac{\sin(\alpha_2 - \alpha_3)}{v_1} + \frac{\sin(\alpha_3 - \alpha_1)}{v_2} + \frac{\sin(\alpha_1 - \alpha_2)}{v_3} = 0 \text{ ..... } 85$$

5642. (E. B. Elliott, M.A.)—If $\frac{x-A}{l} = \frac{y-B}{m} = \frac{z-C}{n}$ be the type of a given doubly-infinite set of straight lines, l, m, n being direction-cosines, and A, B, C given functions of these which make $Adl + Bdm + Cdn$ the exact differential of a function $\phi(l, m, n)$; prove that the polar tangential equation of a class of surfaces cutting all the straight lines orthogonally is $p = \phi(l, m, n) + c$, c being any constant..... 64

5773. (J. L. McKenzie, B.A.)—Through any point P on a circular cubic draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P. ... 54

5846. (W. H. H. Hudson, M.A.)—If the normal at a point P of the lemniscate meet the axis in G, and if GQ, CR be drawn perpendicular to CG, CP respectively to meet the circle described on the line joining the poles of the lemniscate as diameter in Q, R; prove that

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5869. (C. Leudesdorf, M.A.)—ABDC is a square. Find the equation, referred to AB, AC as axes, of the cubic curve having AB, BC, CA, as asymptotes, and having a double point at D; and trace the curve completely. 44

5878. (Prof. Wolstenholme, M.A.) — The four real foci of an axial circular cubic are A, B, C, D, and the foot of the asymptote is O. O is the centroid of A, B, C, D, and if α, β, γ, δ be the algebraical distances OA, OB, OC, OD, the vector equation of the cubic (to A, B, C) is $(\beta^2-\gamma^2)\,AP+(\gamma^2-\alpha^2)\,BP+(\alpha^2-\beta^2)\,CP=0$. Also, if the vertices of the cubic be U, V, W, a Cartesian whose foci are U, V, W will have its vertices at A, B, C, D, and its centre (or triple focus) at O. In a Cassinoid in which, at any point P, $AP \cdot BP = m \cdot OA^2$, if A', B' be the other two axial foci, $A'P \cdot B'P = m \cdot OP^2$, O being the centre. 51

5918. (Prof. Seitz, M.A.)—Prove that the average distance of all points within an ellipse from the extremity of the major axis is

$$\frac{2b\,(1+2e^2)}{3\pi e^2} - \frac{2a\,(1-4e^2)}{3\pi e^3}\sin^{-1}e \text{ ..... } 86$$

6026. (W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T. 39

6027. (Prof. Seitz, M.A.)—Two points are taken at random in a triangle, and a line drawn through them. If two other points be taken at random in the triangle, show that the chance that they will both fall on the same side of the line, is $\frac{11}{18}$. 92

6051. (D. Edwardes.) — A particle describes an ellipse under an attraction to the centre. Prove that (1) the sum of the squares of the velocities at the extremities of a pair of conjugate diameters is constant; (2) the sum of the squares of the reciprocals of the velocities at any two points the directions at which include a right angle is constant. 65

6058. (F. Morley, B.A.) — Two equal perfectly elastic particles fall simultaneously from points in the same vertical line at heights h, h' from a fixed perfectly elastic plane. Show that the height of the point of their first collision from the plane is, x being integral,

$$\frac{[(2x+1)^2h'-h]\,[h-(2x-1)^2h']}{16x^2h'}, \text{ where } h>(2x-1)^2h', \quad <(2x+1)^2h \text{ ....} 75$$

6080. (J. F. Moulton, M.A.)—A quantity of fluid fills a paraboloid of latus rectum c to a height h, the axis being vertical and vertex downwards. The density of the fluid varies as the depth. If the fluid pass into a vessel of the form generated round the axis of x, by the curve $a^4y^2 = 2ch^2x\,(a-x)\,(2a-x)$, where a is any constant, the density will vary as (depth)^2. 88

6102. (The Editor).—If a coin be thrown at random on a horizontal plane; show that (1) the probability of its reaching the plane at an inclination between α and β degrees is $\cos\alpha-\cos\beta$; and (2) the average of all the inclinations with which the coin may reach the plane is the angle subtended by an arc equal to the radius. 50

6123. (Prof. Burnside, M.A.)—If the sides ad, bd, cd of a tetrahedron $abcd$ meet a surface of the second order in the points a, a_1; b, b_1; c, c_1, respectively; prove that the sections of the surface by the planes bca_1, cab_1, abc_1 are all touched by another plane section of the surface..... 84

6150. (H. W. Harris, B.A.)—In the *Quarterly Journal of Mathematics*, Vol. I., Prof. CAYLEY has discussed the locus of the vertex of a triangle circumscribing a given conic and whose vertices move on given curves. In the case of the curves being both conics, the locus is of the eighth degree. Show that, in the case of all three curves being parabolas inscribed in the same triangle, the locus will reduce to a conic; and show how this last is related to the other three. 48

6162. (Prof. Tanner, M.A.) — Required the form of $\phi(x)$ such that $\frac{x\phi'(y) - y\phi'(x)}{x^2\phi(y) - y^2\phi(x)}$ may be a function of xy..... 98

6202. (The Rev. W. A. Whitworth, M.A.) — Give, in as simple a form as possible, the arithmetical rule for finding the seventh root of a given number..... 106

6208. (E. W. Symons, M.A.)—Find the equation of the lines that join the origin to the points of intersection of the curves

$$u_1 + u_2 + \ldots + u_m = 0, \quad v_1 + v_2 + \ldots + v_n = 0. \ldots\ldots 107$$

6236. (W. H. H. Hudson, M.A.) — A triangle ABC, formed of three rods jointed together, is supported by a rough peg under the middle point of AB; prove (1) that the least angle of friction is $\frac{1}{2}(A-B)$, (2) that the sides AC, BC are equally inclined to the vertical, (3) that the strain at the joint C is equally inclined to the horizon with the side AB, and (4) obtain also the magnitude of this and of the other strains..... 83

6254. (J. J. Walker, M.A.) — Find at what points on an ellipse the continued product of the semi-diameter, the perpendicular on tangent, and the cosecant of the angle between them, is a minimum..... 88

6276. (Prof. Tanner, M.A.)—In measuring the distance between two "stations," a surveyor, holding one end of the measuring chain at the near station, places his assistant, who has the other end of the chain, in a direct line between himself and the further station. The assistant leaves a mark from which the operation is repeated, and so on until the line is finished. Suppose that, instead of directing the assistant from the proper place, the surveyor always stands a small distance (h) to the right; find (1) the deviation produced after measuring n chains, the distance between the stations being m chains; and (2) the maximum deviation. 41

6281. (Prof. Wolstenholme, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a parabola passing through the two given foci and having its axis along the given asymptote..... 69

6285. (Prof. Matz, M.A.)—Three points being taken at random in the surface of a circular quadrant; prove that the mean value of all the triangles that can be formed by joining the three points is

$$\frac{r^2}{\pi}\left(\frac{35}{12} + \frac{16}{3\pi} - \frac{131}{3\pi^2}\right). \ldots\ldots 95$$

6292. (J. McDowell, M.A.) — If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \ldots + \frac{1}{n}(-1)^{n-1}p_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}. \ldots\ldots 49$$

6314. (Prof. Seitz, M.A.)—In the surface of a circle two lines are drawn at random in length and direction; prove that the chance that they intersect is $\frac{3}{32}$. 102

6323. (Rev. D. Thomas, M.A.)—If n^m denote the coefficient of x^m in the expansion of $(1+x)^n$, where n is a positive integer and if

$$F_r(n) = \frac{n_r}{1} - \frac{n_{r+1}}{2} + \ldots;$$ prove that $F_r(n) = n_{r-1}\left\{\frac{1}{r} + \ldots + \frac{1}{n}\right\}$ 49

6324. (J. J. Walker, M.A.)—If m, n be positive integers, of which m is not less than n, and p_r, q_r stand for the coefficients of x^r in the development of $(1+x)^n$, $(1-x)^{-m+n-1}$ respectively, prove that

$$\frac{p_1}{q_1} - \frac{p_2}{2q_2} + \frac{p_3}{3q_3} \ldots (-1)^{n-1}\frac{p_n}{nq_n} = \frac{1}{m-n+1} + \frac{1}{m-n+2} + \ldots + \frac{1}{n} \ldots$$ 49

6325, 6367, 6501, 6562. (R. Knowles, B.A., L.C.P.; Rev. D. Thomas, M.A.; and others.)—In Question 6292, prove that

$$p_1 - 2p_2 + \ldots + (n-1)(-)^{n-2}p_{n-1} = 0 \quad \ldots\ldots\ldots\ldots(1);$$

$$\tfrac{1}{2}p_1 - \tfrac{1}{3}p_2 + \ldots + \frac{1}{n+1}(-)^{n-1}p_n = \frac{n}{n+1} \quad \ldots\ldots\ldots\ldots(2);$$

$$\tfrac{1}{3}p_1 - \tfrac{1}{4}p_2 + \ldots + \frac{1}{n+2}(-1)^{n-1}p_n = \frac{n(n+3)}{2(n+1)(n+2)} \ldots\ldots(3);$$

$$1 \cdot 2p_1 - 2 \cdot 3p_2 + \ldots + (n-1)\,n\,(-1)^{n-2}p_{n-1} = 0 \ldots\ldots\ldots(4);$$

$$\frac{p_1}{2 \cdot 3} - \frac{p_2}{3 \cdot 4} + \ldots + \frac{1}{(n+1)(n+2)}(-1)^{n-1}p_n = \frac{n}{2(n+2)} \ldots\ldots(5);$$

also, if p_r denote the coefficient x^r in the expansion of $(1+x)^n$, where n is a positive integer, and if m be also any positive integer, then

$$\frac{p_0}{m+1} - \frac{p_1}{m+2} + \ldots + \frac{(-1)^n p_n}{m+n+1} = \frac{n(n-1)(n-2)\ldots 1}{(m+1)(m+2)\ldots(m+n+1)} \ldots\ldots(6).$$

.....76

6367. (Rev. D. Thomas, M.A.)—See 6325, &c. 76

6389. (J. W. Russell, M.A.)—A conic cuts the sides of a triangle ABC in the pairs of points a_1a_2, b_1b_2, c_1c_2 respectively: if Bb_2, Cc_2 intersect in a_1; Bb_1, Cc_2 in a_2, and so on; and if $\beta_1\beta_2\beta_3\beta_4$, $\gamma_1\gamma_2\gamma_3\gamma_4$ be similarly constructed; show that the straight lines obtained by putting in various suffixes in $A\alpha$, $B\beta$, $C\gamma$ meet three by three in eight points. 50

6394. (W. J. C. Sharp, M.A.)—Find the area contained between an ordinary catenary, the tractrix which is its involute, and any tangent to the first..... 93

6405 & 6531. (Prof. Sylvester, F.R.S.)—If p, q, r, s are the distances of a point in a circular cubic from the four concyclic foci A, B, C, D; prove (1) that

$$\frac{(p-q)(q-r)(r-p)}{ABC} = \frac{(q-r)(r-s)(s-q)}{BCD} = \frac{(r-s)(s-p)(p-r)}{CDA} = \&c.;$$

also prove (2) that only one proper circular cubic can be drawn having four concyclic foci at the angles of a trapezoid. 46

6421. (W. H. H. Hudson, M.A.)—If a point P moves so that SP . PM is constant, where S is a fixed point and PM is perpendicular to a fixed line, and if tangents at any two points of its path be drawn and produced to meet the asymptote, prove that the triangle included between the two tangents and the asymptote is equal to the area included between them and the corresponding arc of the locus of the extremity of the polar subtangent..... 43

6426. (C. Leudesdorf, M.A.)—Prove that, if n be odd, the product of the squares of the differences of the roots of the equation

$$x^n + nx^{n-1} + (n-1)^{n-1} = 0 \quad \text{is} \quad 2n^n (n-1)^{(n-1)^2} \ldots\ldots 40$$

6436. (D. Edwardes.) — If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle lie in a straight line. 78

6440. (Prof. Townsend, F.R.S.)—The slab of the table in Question 6226 being supposed elliptical in place of rectangular; determine, all other particulars remaining unaltered, the slab of maximum area in its critical position, and the lengths of its semi-axes a_1 and b_1 in terms of the coordinates a and b of the post. 23

6464. (W. S. B. Woolhouse, F.R.A.S.)—Show that

$$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2 \cdot \frac{2 \cdot 4 \ldots 2n}{3 \cdot 5 \ldots 2n+1} (a+x) \left\{ a^n \left(1 + \frac{a}{x}\right)^{-\frac{1}{2}} \text{as far as } x^n \right\} \ldots 42$$

6471. (Prof. Crofton, F.R.S.)—See 5388 & 6471. 25

6472. (Prof. Townsend, F.R.S.) — Two normals to an ellipse being supposed to intersect at right angles on the curve; required the limits to their ratio requisite to their being both maxima vectors from their point of intersection to the curve. 110

6475. (The late Prof. Clifford, F.R.S.) — Let U, V be any two cubic functions of x; show that a quartic function $f(x)$ may always be found such that, by the substitution $y = U : V$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $M\,dy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant. 21

6477. (Prof. Matz, M.A.)—Two points are (1) taken at random on the *arc*, and another point in the *surface* of a circular quadrant; and again, two points are (2) taken at random in the *surface*, and another point on the *arc* of a circular quadrant; prove that the average areas of the triangles formed by joining these points in their respective order will be to each other as $45\pi^2 + 84\pi - 684 : 35\pi^2 + 64\pi - 524$ 96

6482. (The Editor.)—Within a tetrahedron ABCD a point O is taken, and straight lines AOE, BOF, COG, DOH are drawn to meet the faces in E, F, G, H; find the average volumes of the tetrahedron EFGH..... 30

6485. (Elizabeth Blackwood.) — Given that each of the compound events *axy*, *byz*, *czx* is always followed by two at least of the events *d*, *e*, *f*, and that each of the compound non-occurrences *d'x'y'*, *e'y'z'*, *f'z'x'* implies the non-occurrence of two at least of the events *a*, *b*, *c*; what conclusion may thence be drawn with reference to the occurrence or non-occurrence of the events *a*, *b*, *c*, *d*, *e*, *f*, without mentioning the events *x*, *y*, *z*? 24

6501. (R. Knowles, B.A.)—See 6325, &c. 76

6504. (Prof. Townsend, F.R.S.)—A system of smooth equal spheres, two terminal and fixed, and the remainder in contact each with the two adjacent, being supposed to form an arch in a vertical plane, in unstable equilibrium under the action of gravity; determine, given all particulars, the ultimate form of the arch, when the component spheres are indefinitely increased in number and diminished in magnitude. 27

6506. (Prof. Crofton, F.R.S.)—Express the sum of the series (n being integral) $S = x^n + n^2 x^{n-1} + \frac{n^2(n-1)^2}{1.2} x^{n-2} + \&c.$ 99

6509. (Prof. Genese, M.A.)—Prove (1) that the directrices of a conic are common chords of the conic and its director circle; (2) that they are also the radical axes of that circle and of point-circles at the foci. 26

6512. (Prof. Matz, M.A.)—A heavy prismatic bar of infinitesimal cross-section and length $2c$ rests against the concave arc of a vertical elliptic bowl and a pin placed at the focus; prove that the bar is in equilibrium when its inclination to the vertical is $\cos^{-1}\left\{\frac{1}{e}\left[\frac{b}{a^{\frac{1}{2}}c^{\frac{1}{2}}} - 1\right]\right\}$, where (a, b) are the axes and e the eccentricity of the ellipse..... 89

6516. (C. W. Merrifield, F.R.S.) — Prove that the remainders obtained by "casting out the nines" from the terms of any geometrical expression recur in regular order. 30

6522. (W. R. Westropp Roberts, M.A.)—A sphere is gently placed on a rough inclined plane of given height and length, and descends under the influence of gravity; prove that the value of the coefficient of friction for which the *vis viva* gained in the descent is a minimum is

$$2mgh \times \frac{2a}{a + (a^2 + k^2)^{\frac{1}{2}}} \text{.....} 42$$

6523. (G. F. Walker, M.A.) — Three quantities lie between $(0, a)$, $(0, b)$, $(0, c)$ respectively; find the chance that their product is less than d^3 $(d^3 < abc)$. 64

6525. (H. McColl, B.A.) — If n quantities be each taken at random between a and $-a$, show that the chance that their sum will be between b and $-b$ is

$$1 - \frac{2}{(2a)^n n!}\left\{(na-b)^n - n(na-b-2a)^n + \frac{n(n-1)}{1.2}(na-b-4a)^n - \frac{n(n-1)(n-2)}{1.2.3}(na-b-6a)^n + \ldots\right\} \&c.,$$

the series to be continued as long as $na-b-2ra$ is positive..... 90

6526. (The Editor.) — From the points in which any two conjugate diameters of an ellipse meet a fixed tangent, tangents are drawn to a fixed confocal; prove that the locus of their intersection is a circle whose centre is on the normal to the fixed tangent. 80

6531. (Prof. Sylvester, F.R.S.)—See 6405 & 6531 46

6532. (Professor Sylvester.)—Prove that the condition of an equation of the 5th degree being linearly transformable into a recurring form is $I = 0$, where, $\alpha, \beta, \gamma, \delta, \epsilon$ being the 5 roots, I is the product of the 15 distinct factors obtainable from the permutations of

$$\pm[(\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma) + (\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)];$$

and show that I is a symmetrical function of the 18th order in the coefficients of the given equation. 32

6533. (Prof. Crofton, F.R.S.)—Prove that

$$(r+1)^r = r^r + 2^0 \cdot r(r-1)^{r-1} + 3^1 \cdot \frac{r(r-1)}{1.2}(r-2)^{r-2} + 4^2 \cdot \frac{r(r-1)(r-2)}{1.2.3} \times (r-3)^{r-3} + 5^3 \cdot \frac{r(r-1)(r-2)(r-3)}{1.2.3.4}(r-4)^{r-4} + \ldots + (r+1)^{r-1} \text{.....} 60$$

6534. (Prof. Salmon, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that $aU_1+bU_2+cU_3+dU_4+eU_5$ may be either a perfect square L^2 or the product of two factors MN. Prove that the line L touches a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point. 22

6538. (Prof. Genese, M.A.) — Find the envelop of the latera recta of conics having a given director circle, and passing through a given point. 85

6539. (Prof. Minchin, M.A.)—Prove the following fundamental theorem:—In all motion of a rigid body parallel to one plane, there is at every instant a point which has no acceleration (let us call it the *instantaneous acceleration centre*, and denote it by J); the acceleration of every point in the body is directly proportional to its distance from J, and its direction makes a constant angle with the line joining the point to J... 44

6540. (Prof. Matz, M.A.)—A, B, C, D play a set of games in which two are partners against the other two, any two being equally likely to be partners, and drawn games being impossible. If A's chance of winning a single game with B, C, D for partner be p, q, r respectively; find the chances that A, B, C, D respectively will win a out of $a+b$ games..... 63

6541. (Prof. Nash, M.A.) — A cardioid being generated by a circle rolling upon a fixed equal circle; prove that the polar reciprocal of the cardioid with respect to the fixed circle is a nodal circular cubic; that there are two real axial foci besides the double focus at the node; and that the vector equation is $3\rho_1 \pm 2\rho_2 - \rho_3 = 0$, where ρ_2 is the distance from the double focus. 65

6545. (Elizabeth Blackwood.)—In the integral

$$\int dw \int dx \int dy \int dz\, \phi\,(w, x, y, z)$$

the four variables are each restricted between the limits a and $-a$, and the integration is further restricted by the condition $w+x>yz$; find the limits of integration when z varies first, y next, and w last..... 61

6546. (Professor Purser, M.A.)—If the equation of a surface be expressed in the form $p=F(\theta, \phi)$, where p is the perpendicular from a fixed point on the tangent plane, θ, ϕ the usual angles determining its position; show that the sum of the principal radii of curvature is given by $R+R'=\Delta . p$, where Δ represents the differential operator

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\phi^2}+2,$$

corresponding to the equation of LAPLACE's functions of Order 1. Hence, employing BOOLE's form of solutions, it appears that the general equations of surfaces having their principal curvatures equal and opposite is

$$p=\frac{d}{d\theta}[\sin\theta F_1(e^{i\phi}\tan\tfrac{1}{2}\theta)+\sin\theta F_2(e^{-i\phi}\tan\tfrac{1}{2}\theta)]. \ldots\ldots 33$$

6547. (E. B. Elliott, M.A.) — From a theorem in plane geometry another is obtained by inversion with regard to any circle in the plane; and from the two theorems, by stereographic projection on a sphere having for centre and radius the centre and radius of inversion, two theorems of spherical geometry are obtained. Prove that these two are identical. ... 97

6548. (W. J. C. Sharp, M.A.)—If the normals at four points on an ellipse meet in a point, prove that the sum of the excentric angles at these points is an odd multiple of π. 62

6549. (Rev. W. A. Whitworth, M.A.) — If (3N)! be divided by e (the base of Napierian logarithms), prove that the integer nearest to the quotient will be a multiple of 3N − 1. 32

6550. (W. R. Westropp Roberts, M.A.) — A circular elastic lamina, of uniform thickness and isotropic material, is capable of motion round an axis passing through its centre and perpendicular to its plane; being given the strength of the material, determine the angular velocity which is just sufficient to produce permanent alteration of form..... 67

6551. (R. A. Roberts, M.A.)—If three lines, making an angle θ with a conic at points whose eccentric angles are α, β, γ, meet in a point, show that $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = \frac{2ab}{a^2-b^2}\cot\theta$. 39

6552. (J. J. Walker, M.A.) — If $u = x^n \log x$, prove that there is a form of $\frac{d^r u}{dx^r}$ other than that given by LEIBNITZ's Theorem; viz.,

$$n\,.\,n-1\,\ldots\,n-r+1\,.\,x^{n-r}\left(\log x+\frac{1}{n-r+1}+\frac{1}{n-r+2}\,\ldots\,+\frac{1}{n}\right).\ldots\ 72$$

6553. (G. F. Walker, M.A.)—Solve the equations

$$x^2(y+z) = a^3,\quad y^2(z+x) = b^3,\quad z^2(x+y) = c^3. \ldots\ 86$$

6554. (C. E. Bickmore, M.A.)—Solve the equations

$$xyz+x-y-z = a(1+yz-zx-xy),\quad xyz+y-z-x = b(1+zx-xy-yz),$$
$$xyz+z-x-y = c(1+xy-yz-zx). \ldots\ 53$$

6556. (C. Leudesdorf, M.A.)—If $3nz^2 = mx^2-2lxy$, $3n^2z = l^2y+2lmx$, $n^3 = l^3m$, prove that $z^3 = x^2y$..... 91

6560. (W. B. Grove, B.A.)—In a certain school, every one who learns both English and French, or neither of them, does not learn Arithmetic; every one who learns Arithmetic learns both English and German or neither of them; everyone who learns French but not German learns either English or not Arithmetic. Assuming all combinations not inconsistent with these data to exist, prove (1) that the number of those who learn Arithmetic does not exceed the number of those who learn English; and (2) that the above conditions may be replaced by a single simple but exactly equivalent one. 29

6561. (E. W. Symons, M.A.) — A conic is self-conjugate to the triangle of reference; (x', y', z') are the areal coordinates of its centre, and (x, y, z) the distances of this point from the vertices of the triangle; prove that the radius of the director circle is $(x'x^2+y'y^2+z'z^2)^{\frac{1}{2}}$. 101

6562. (R. Knowles, B.A.)—See 6325, &c. 76

6563. (Prof. Sylvester, F.R.S.) — 1. The velocities of three bodies in the direction of the lines joining them with their centre of gravity, and the lengths of these lines, being given; find the transverse velocities [that is, perpendicular respectively to the former ones] in the plane of the bodies, in order that the centre of gravity, and the principal axes through it of the system, may remain at rest for a moment of time.

2. Given the distances of three bodies from their centre of gravity, determine the angles which they make respectively with the principal axes of the system, and the values of the moments of inertia in respect to these axes. 55

6565. (Prof. Minchin, M.A.)—Prove that the areas of roulettes, in their most general forms, follow the law of circular transformation obtained by STEINER, for the areas of pedals (see WILLIAMSON'S *Integral Calculus*, 3rd ed., p. 202); and thus, in virtue of a general kinematical theorem, deduce at once KEMPE'S theorem (*ibid.* p. 210). 59

6569. (J. R. Harris, M.A.)—Two trains, of lengths m, n respectively, are at distances a, b from a level crossing, towards which they are moving with velocities that are equally likely to be any possible magnitude; prove that, according as the tail of the second train is nearer to or further from the crossing than the head of the first train, the chance of an accident is $(a > b)$ $\frac{1}{2}\frac{an+bm+mn}{a(a+m)}$, or $1-\frac{1}{2}\frac{a^2+b^2+am+bn}{(a+m)(b+n)}$. 82

6579. (J. J. Walker, M.A.)—If P, P′ are two points on an ellipse, the foci of which are S, H: and if X, X′, Y are any points in the productions of SP, SP′, PP′ respectively; prove that

$\cos\frac{1}{2}(XPY+X'P'Y) : \cos\frac{1}{2}(HPY+HP'Y) = \cos\frac{1}{2}PSP' : \cos\frac{1}{2}PHP$... 77

6581. (W. R. Westropp Roberts, M.A.)—Through a fixed chord AB of a twisted cubic are drawn two planes harmonically conjugate with two fixed planes through the same chord. Show that, if the variable planes meet the cubic in two points P and Q, the chord PQ meets two fixed lines. 79

6585. (R. E. Riley, B.A.)—Any chord AB is drawn through A, a fixed point of a circle ABC, and a circle CDE is described touching ABC, and the chord AB at its middle point D; prove that the centre of CDE lies on a cardioid, whose pole is the centre of ABC. 92

6586. (D. Edwardes.)—A string of length $2l$ is suspended from two points in the same horizontal line distant $2a$ from each other. If the distance between the points be slightly increased by the quantity 2δ, prove that the vertex of the catenary will ascend through the space $\frac{z^2+cz-l^2}{cl-az-ae}.\delta$, where c is tension at lowest point, and z the depth of lowest point below the line of suspension. 74

6588. (A. Martin, M.A.)—A person is to draw two from the tickets 1, 2, 3 ... 100, all tickets being equally likely. If both tickets be squares, he is to receive £100; if the lower, only £50; if the higher, only £20; find the worth of his chance of gain. 90

6590. (E. B. Elliott, M.A.)—If there be any distribution of mass M in space, if I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, and A and B respectively; prove that the moment of inertia about any fourth parallel line may be expressed in the form

$$xI_a+yI_b+zI_c-M(a^2yz+b^2zx+c^2xy),$$

x, y, z being coordinates whose sum is unity of the fourth line with regard to the prism formed by the three. 75

6591. (W. H. H. Hudson, M.A.)—A man standing at the North Pole whirls 24 lbs. Troy weight on a smooth horizontal plane, by a string a yard long, at the rate of 100 turn a minute, and finds that the difference of the forces he has to exert, according as he whirls it one way or the opposite, is roughly 39 grains: find the period of the earth's rotation. 87

6595. (The late S. M. Drach, F.R.A.S.)—
If six-digit number N be so reckoned
That twice (6th less 3rd) plus thrice (5th less 2nd),
With 4th less 1st, announce sum sevenfold,
Whole number N by *sevens* can be told.
Quadruple (6th less 3rd) plus thrice (5th less 2nd)
With 1st less 4th, is test if *thirteen*'s beckoned. 71

6596. (Prof. Sylvester, F.R.S.)—APB is a semi-ellipse of which S is the focus, and AB the major axis; and AMB is a semicircle drawn on AB; show (1) that if two bodies, attracted each towards its own centre of force at S, and leaving the point A at the same moment, move in these two curves, the one will be at P when the other is at M, PM being any ordinate perpendicular to AB; and (2) state also the theorem or the case when S coincides with A. 79

6599. (Prof. Minchin, M.A.)—At any two points, A and B, in the plane of any lamina, draw lines AO and BO equal to the radii of gyration of the lamina round A and B respectively, thus constructing a triangle ABO; then prove that the radius of gyration round any point P in base AB is PO. 83

6603. (Prof. Genese, M.A.)—Given the base and vertical angle of a triangle, find the envelop of the nine-point circle. 86

6605. (Prof. Purser, M.A.)—Four bars are jointed together at the ends, two of them crossing each other; show that, when the four ends lie on the same circle, the difference of the areas of the triangles formed by the bars is less than in any other position. 99

6606. (Elizabeth Blackwood.)—In the integral

$$\int dw \int dx \int dy \int dz\, \phi\,(w, x, y, z),$$

the variables are each between the limits a and $-a$, and the integration is further restricted by the condition that $(wx-yz)$ is positive; determine the limits of integration. 81

6608. (The Editor.)—Find, to four places of decimals, and, if possible, in a finite form, the value of the following series and its equivalent integral, deduced in Prof. Spitz's solution of the Editor's Question 6482 (*Reprint*, Vol. XXXV., p. 30):—

$$\frac{1}{2^2}-(1-\tfrac{1}{2})\frac{1}{3^2}+(1-\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^2}-(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})\frac{1}{5^2}+\ldots \equiv \int_0^1 \frac{\log(1-x)}{1+x}\log x\,dx.$$

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6610. (The Rev. T. P. Kirkman, M.A., F.R.S.)
To a wee planetoid, but recently out,
I am bound to dispatch an opinion,
On how to effect a design they're about
Of improving their little dominion.
Tired of their islands, they long for a Continent:
Here is the statement they give me, their want anent:—

Their cities line their woody shore,
And creeks, that curl around in curious ways,
Beating the bays
Of Hellas or of Scotland;
So that of land is little more
About them than of not-land.

To no one city can there be
Of home approaches more than sacred three.
Each has two shore lines, and the only other
Is ferry o'er the creek to fronting town,
Or high-road, bound of province, to another,
Washed by another inlet. Peer and clown,
The laws from olden time forbidding byroad,
Must travel by that shore, or boat, or high-road.
Yet can two cities, ferry-bound,
Have commerce over ground
Along the winding shore-line round.

And there are towns on every strand
Without or ferry or a road inland,
Which, two and two on different isles, are able
To speak through ocean by electric cable.
These triad lines must be,
Unchanged for ever, bounds of earth and sea.

The ferries or the cable-lines between,
Some lordships of their shallow floods they mean
To bank and raise, the while they excavate
Along high road or roads, and inundate
What provinces they must, to make one strand,
And every cable a ferry, or edge of land.
How to contrive that every town may keep
Its own third line with piers along the deep?

To him who can tell them they give the remar-
Kable profit of purchasing shares at par;
And the Council will make him an F.S.R
On islands, three, four, or two,
Towns, to threescore or two,
Cover with triedral summits your n-edron;
When they are penned, run
Over your islands a pencilling cloud,
Giving the cities the shore-lines allowed.
The white will all be
Ferry, cable, and sea.
You may feel a bit proud,
If, after some labour, you find what you want, an ent-
Ire single circle of towns on a Continent.
When it is found, there is nothing to face
But proof of a rule to fit every case.

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6617. (The Rev. W. A. Whitworth, M.A.)—A row of n letters may be permuted by moving any letter backwards or forwards *over the next two letters*. Show that, by continually repeating this operation, the n letters can be brought into $\frac{1}{2}n!$ different orders; that is, exactly half of the possible permutations of the n letters can be formed..... 83

6620. (G. F. Walker, M.A.) — Prove that the line of quickest descent, from one to the other of two confocal ellipses whose major axis is vertical, subtends equal angles at the foci. 108

6622. (T. H. Attwater, M.A.)—If the tangent to an ellipse at P meet the major axis in T, and the tangent at P′ meet the minor axis in T′, and if CP, CP′ are conjugate; prove that TT′ is parallel to an equi-conjugate diameter..... 90

6634. (Elizabeth Blackwood.)—A straight line is divided into n random segments, show that the chance that these segments can form an n-sided polygon is $1-n2^{1-n}$. 111

6638. (H. G. Dawson, M.A.)—If four of the roots of

$$(a, b, c, d, e, f)(x, 1)^5 = 0$$

be connected by the relation $\alpha+\beta=\gamma+\delta$, prove that the remaining root ϵ is given by the equations $z=a\epsilon+b$, $z^3-8\mathrm{H}z+16\mathrm{G}=0$, where $\mathrm{H}=b^2-ac$, $\mathrm{G}=2b^3-3abc+a^2d$. 102

6639. (J. J. Walker, M.A.) — The constants a, b, c of an empirical formula $p=a+b\tan(\theta+c)$ are to be determined from three pairs of observed simultaneous values of the variables, viz., $p\theta$, $p'\theta'$, $p''\theta''$; prove that (1) the single real value of c is that given by

$$\tan c = \Sigma\, p\cos\theta\sin(\theta'-\theta'') \div \Sigma\, p\sin\theta\sin(\theta'-\theta'');$$

and (2), if $(p'-p'')^2\cos(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta') = q$....,

$$a = (pq+p'q'+p'q'') \div (q+q'+q''),$$

$$b = -(p'-p'')(p''-p)(p-p')\sin(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta') \div (q+q'+q'').$$

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6641. (W. S. McCay, M.A.) — Prove that the quadrilinear coordinates of the focus of the parabola touching four lines x_1, x_2, x_3, x_4 are given by the equations $\mathrm{R}_1x_1=\mathrm{R}_2x_2=\mathrm{R}_3x_3=\mathrm{R}_4x_4$, where R_1 is the radius of the circle circumscribing the triangle $x_2x_3x_4$, &c. 104

6645. (E. B. Elliott, M.A.)—Prove that

$$\int_0^a x^n(2a-x)^n\,dx = 2^{2n}\int_0^a x^n(a-x)^n\,dx \quad \ldots\ldots (1),$$

$$\frac{2^{2n}-1}{\Gamma(2n+2)} = \frac{1}{\Gamma(2n+1)} + \frac{1}{1.2}\cdot\frac{1}{\Gamma(2n)} + \frac{1}{1.2.3}\cdot\frac{1}{\Gamma(2n-1)} \ldots (2);$$

the series stopping at the n^{th} term if n be a positive integer or zero... 105

6646. (W. B. Grove, B.A.) — There are 24 cards, every one of which is painted with one or more of the three colours—red, blue, and yellow. Observation informs us that there are 15 painted with blue, 19 with yellow, and 16 not painted with red; 8 painted with both red and blue, 3 with both red and yellow, 10 with both blue and yellow. Find the law or laws which regulate the combinations of the colours. 103

6656. (W. H. Besant, F.R.S.) — Find (1) the volume, (2) the surface, generated by the revolution about the initial line of the curve $r=a(1+\cos\theta)$; also (3) the mean value of the radii to all points within the volume enclosed; and prove (4) that the mean value of the radii to the surface is equal to a when the radii are drawn symmetrically in space, and to $\frac{1}{4}\pi a$ when the surface is covered equably by the ends of the radii; also (5) that, if the volume be made up of a series of shells of equal volumes, bounded by a succession of surfaces similar and similarly situated to the outer surface, and having the same cuspidal point, the mean value of their surfaces is three-fifths of the outer surface. 103

6658. (Prof. Crofton, F.R.S.)—Solve the functional equations

$\phi(ax+b) = 1+\phi(x)$, $\phi(x^n) = 1+\phi(x)$, $\phi\left(\frac{1}{1-x}\right) = 1+\phi(x)$. ... 116

6677. (Dr. Macfarlane, F.R.S.E.)—(1) A lady, on being asked about a photograph in her album, gave the following answer:—"You know that I have no daughters; that person's daughter's son was the father of a grandchild of mine." [Also solve, by a like method, the two following problems:—

(2) "If Dick's father is Tom's son, what relation is Dick to Tom?"
(3) "Sisters and brothers I have none, but that man's father is my father's son."]
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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6475. (By the late Prof. CLIFFORD, F.R.S.)—Let U, V be any two cubic functions of x; show that a quartic function $f(x)$ may always be found such that, by the substitution $y = U : V$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $M\,dy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant.

Solution by ROBERT RAWSON.

Let $y = \frac{U}{V}$; then $V^2 dy = \left(V\frac{dU}{dx} - U\frac{dV}{dx}\right)dx = D\psi(x)\,dx$ ..... (1, 2).

But $$\frac{dx}{\{f(x)\}^{\frac{1}{2}}} = \frac{D_1 V^2 \phi(x)}{\{D_1^2 \phi(x)^2 . x-\alpha_1 . x-\alpha_2 . x-\alpha_3 . x-\alpha_4\}^{\frac{1}{2}}} \cdot \frac{dy}{D\psi(x)},$$

$$\therefore \frac{dx}{(x-\alpha_1 . x-\alpha_2 . x-\alpha_3 . x-\alpha_4)^{\frac{1}{2}}} = \frac{D_1\,dy}{D(y-\beta_1 . y-\beta_2 . y-\beta_3 . y-\beta_4)} \ldots (3),$$

if $\phi(x) = \psi(x)$ and $D_1^2\phi(x)^2 . x-\alpha_1 \ldots x-\alpha_4 = U-\beta_1 V \ldots U-\beta_4 V \ldots$ (4, 5).

It remains, therefore, to satisfy the equations (4) and (5), for any given rational functions of U and V in order that equation (3) may exist.

The function $\psi(x)$ is of $(2n-2)$ dimensions in x, when U and V are of (n) dimensions in x (See CAYLEY'S *Elliptic Functions*, page 163). By the Question, U and V are cubic functions.

Putting (6), $y = \frac{A(x^3 - px^2 + qx - r)}{x^3 - p_1 x^2 + q_1 x - r_1}$, we have

$$\psi(x) = x^4 - \frac{2(q-q_1)}{p-p_1}x^3 + \frac{3r-3r_1+p_1 q-pq_1}{p-p_1}x^2 - \frac{2(p_1 r - pr_1)}{p-p_1}x + \frac{q_1 r - qr_1}{p-p_1} \ldots\ldots\ldots\ldots (7),$$

$$= x-\nu_1 . x-\nu_2 . x-\nu_3 . x-\nu_4 = \phi(x),$$

where $\nu_1, \nu_2, \nu_3, \nu_4$ are the roots of (7) and $D = p - p_1$.

Equation (5) then becomes

$$D_1^2 (x-\nu_1)^2 \ldots (x-\nu_4)^2 . x-\alpha_1 \ldots x-\alpha_4 = [A(x^3 - px^2 + qx - r) - \beta_1(x^3 - p_1 x^2 + q_1 x - r_1)] \ldots [A(x^3 - px^2 + qx - r) - \beta_4(x^3 - p_1 x^2 + q_1 x - r_1)] \ldots (8),$$

or $$(x-\nu_1)^2 \dots (x-\nu_4)^2 \cdot x-\alpha_1 \dots x-\alpha_4$$

$$= \left(x^3 - \frac{Ap-\beta_1 p_1}{A-\beta_1} x^2 + \frac{Aq-\beta_1 q_1}{A-\beta_1} x - \frac{Ar-\beta_1 r_1}{A-\beta_1}\right) \dots$$

$$\dots \left(x^3 - \frac{Ap-\beta_4 p_1}{A-\beta_4} x^2 + \frac{Aq-\beta_4 q_1}{A-\beta_4} x - \frac{Ar-\beta_4 r_1}{A-\beta_4}\right) \quad \dots(9),$$

where $$D_1^2 = A-\beta_1 \cdot A-\beta_2 \cdot A-\beta_3 \cdot A-\beta_4 \quad \dots\dots\dots\dots(10).$$

Equation (9) may be transformed as follows by the relation

$$x^3 - ax^2 + bx - c = \left\{x - \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \cdot \left\{x - \tfrac{1}{3}\left[a - 2\,(a^2-3b)^{\frac{1}{2}}\right]\right\} \dots(11),$$

if $$\left\{\tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \cdot \left\{\tfrac{1}{3}\left[a - 2\,(a^2-3b)^{\frac{1}{2}}\right]\right\} = c \dots\dots\dots\dots(12).$$

Putting
$$\left.\begin{aligned} a &= \frac{Ap-\beta_1 p_1}{A-\beta_1}, \quad b = \frac{Aq-\beta_1 q_1}{A-\beta_1}, \quad c = \frac{Ar-\beta_1 r}{A-\beta_1} \\ &\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ a_3 &= \frac{Ap-\beta_4 p_1}{A-\beta_4}, \quad b_3 = \frac{Aq-\beta_4 q_1}{A-\beta_4}, \quad c_3 = \frac{Ar-\beta_4 r}{A-\beta_4} \end{aligned}\right\} \dots\dots\dots(13),$$

(9) becomes
$$\left.\begin{aligned} &(x-\nu_1)^2 \dots (x-\nu_4)^2 \cdot x-\alpha_1 \dots x-\alpha_4 \\ &= \left\{x - \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \left\{x - \tfrac{1}{3}\left[a - 2\,(a^2-3b)^{\frac{1}{2}}\right]\right\} \dots \\ &\dots \left\{x - \tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right]\right\}^2 \left\{x - \tfrac{1}{3}\left[a_3 - 2\,(a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} \end{aligned}\right\} \dots(14),$$

with the four conditional equations

$$\left\{\tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \left\{\tfrac{1}{3}\left[a - 2\,(a^2-3b)^{\frac{1}{2}}\right]\right\} = c, \dots,$$

$$\left\{\tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} \left\{\tfrac{1}{3}\left[a_3 - 2\,(a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} = c_3 \dots\dots\dots(15).$$

From (14), it readily follows that

$$\nu_1 = \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right], \dots \nu_4 = \tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right],$$

and $$\alpha_1 = \tfrac{1}{3}\left[a - 2\,(a^2-3b)^{\frac{1}{2}}\right], \dots \alpha_4 = \tfrac{1}{3}\left[a_3 - 2\,(a_3^2-3b_3)^{\frac{1}{2}}\right] \dots\dots(16).$$

There are twelve equations—four in (15) and eight in (16)—to be satisfied by the quantities A, p, q, r, p_1, q_1, r_1, $\alpha_1 \dots \alpha_4$, $\beta_1 \dots \beta_4$, leaving only three arbitrary constants out of the fifteen in order to establish the differential equation

$$\frac{dx}{(x-\alpha_1 \dots x-\alpha_4)^{\frac{1}{2}}} = \frac{(A-\beta_1 \dots A-\beta_4)^{\frac{1}{2}}\, dy}{(p-p_1)\,(y-\beta_1 \dots y-\beta_4)^{\frac{1}{2}}} \quad \dots\dots\dots(17).$$

To reduce (17) to the standard form, it is necessary to put $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = \frac{1}{k}$, $\alpha_4 = -\frac{1}{k}$, $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = \frac{1}{k'}$, $\beta_4 = -\frac{1}{k'}$.

The systems (15), (16) may be written as follows:—

$$V_1^2 a_1 = c, \dots V_4^2 a_4 = c_3 \quad \dots\dots\dots(18),$$

$$2V_1 = p - a_1 \dots 2V_4 = p - a_4, 3a_1 = a - 2\,(a^2-3b)^{\frac{1}{2}} \dots 3a_4 = a_3 - 2(a_3^2-3b_3)^{\frac{1}{2}} \dots(19).$$

6534. (By Prof. SALMON, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that $aU_1 + bU_2 + cU_3 + dU_4 + eU_5$ may be either a perfect square L^2 or the

product of two factors MN. Prove that the line L touches a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

Solution by J. J. WALKER, M.A.

The lines M, N being $lx+my+nz$, $l'x+\ldots$, and f an indeterminate multiplier, the six equations

$$a_1a+a_2b+\ldots+a_5e+ll'f=0, \qquad 2f_1a+2f_2b+\ldots+(mn'+m'n)f=0,$$
$$b_1a+b_2b+\ldots+b_5e+mm'f=0, \qquad 2g_1a+2g_2b+\ldots+(nl'+ln')f=0,$$
$$c_1a+c_2b+\ldots+c_5e+nn'f=0, \qquad 2h_1a+2h_2b+\ldots+(lm'+ml')f=0$$

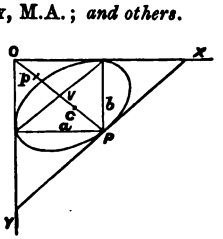
must hold. Eliminating $a \ldots f$ among these, the result is

$$All'+Bmm'+Cnn'+F(mn'+nm')+G(nl'+ln')+H(lm'+ml')=0,$$

where A ... H are the minors of the determinant of the system of six equations formed by omitting the last column with the first, second ... sixth rows successively. But the relation among $l, m, n, l' \ldots$ thus found is the condition that (MN) should be conjugate with respect to the conic $(BC-F^2)x^2+\ldots+2(FG-CH)xy=0$; so that, if M be given, N passes through the given point which is the pole of M with respect to this conic. As a particular case, if MN coincide in one line L, this must touch the same conic. In fact the relation then becomes $Al^2+\ldots+2Hlm=0$.

6440. (By Prof. TOWNSEND, F.R.S.)—The slab of the table in Question 6226 being supposed elliptical in place of rectangular; determine, all other particulars remaining unaltered, the slab of maximum area in its critical position, and the lengths of its semi-axes a_1 and b_1 in terms of the coordinates a and b of the post.

I. *Solution by* R. E. RILEY, B.A.; J. A. KEALY, M.A.; *and others.*

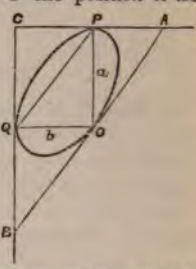


Going on the lines of the solution to Question 6226, we see that the ellipse must be the maximum ellipse inscribed in the triangle formed by the two lines and its tangent at the fixed point when in the critical position. Its area is therefore $\frac{2\pi ab}{3\cdot 3^{\frac{1}{2}}}$. Also O is a point on the director circle, and CP = 2CV, OC = $\frac{2}{3}$OP; and therefore $a_1^2+b_1^2=\frac{4}{9}(a^2+b^2)$, and $a_1b_1=\frac{2ab}{3\cdot 3^{\frac{1}{2}}}$, from which a_1 and b_1 can be found.

[This solution, though satisfactory so far as it goes, is incomplete, since the two normals a, b are not *always* the two *maxima* vectors of the ellipse from P.]

II. *Solution by the* PROPOSER.

Let CA, CB be the walls of the room, and O the position of the post; then, drawing through O the perpendiculars OP and OQ to the walls, and the parallel AB to the connector PQ of their feet P and Q, the ellipse OPQ, inscribed to the triangle ABC at the points of bisection O, P, Q of its sides, will, within certain limits of the ratio of $a : b$, be the contour of the required slab in its critical position.



For, the ellipse in question being that of greatest area that could be inscribed in the triangle ABC, no ellipse of greater area could under any circumstances attain to, much less pass through, the position in which its tangent at O would coincide with AB; and, that position being supposed attained in any manner by the ellipse OPQ, regarded as a rigid lamina movable in the plane of the figure; then, as, within certain limits of the ratio of $a : b$, OP and OQ are at once the two maxima vectors from O to the curve and the two minima vectors from O to the walls, a small rotation in either direction of the lamina round O as centre would, as in the case of the rectangular slab, disengage simultaneously its two points P and Q from contact with the walls, and extricate it consequently from its critical position.

To find the values of a_1 and b_1 in terms of a and b. Since, from the geometry of the ellipse, as may be shewn without difficulty,

$$a_1^2 + b_1^2 = \tfrac{4}{9}(a^2 + b^2), \text{ and } a_1^2 b_1^2 = \tfrac{4}{27} a^2 b^2 ;$$

the values of a_1^2 and b_1^2 are consequently the roots of the quadratic equation

$$u^4 - \tfrac{4}{9}(a^2 + b^2)\, u^2 + \tfrac{4}{27} a^2 b^2 = 0,$$

and therefore $= \frac{2}{9}\left[(a^2 + b^2) \pm (a^4 + b^4 - a^2 b^2)^{\frac{1}{2}}\right]$, respectively.

To find the limits of the ratio of $a : b$ requisite to the validity of the above solution. The radii of curvature of the ellipse at P and Q, as may also be seen without difficulty, being respectively $\frac{1}{2}\frac{b^2}{a}$ and $\frac{1}{2}\frac{a^2}{b}$, in order that they should be less than a and b respectively, and therefore that OP and OQ should be both maxima vectors from O to the curve, we must have at once $b^2 < 2a^2$ and $a^2 < 2b^2$; and therefore, &c.

6485. (By ELIZABETH BLACKWOOD.)—Given that each of the compound events axy, byz, czx is always followed by two at least of the events d, e, f, and that each of the compound non-occurrences $d'x'y'$, $e'y'z'$, $f'z'x'$ implies the non-occurrence of two at least of the events a, b, c; what conclusion may thence be drawn with reference to the occurrence or non-occurrence of the events a, b, c, d, e, f without mentioning the events x, y, z?

Solution by H. McColl, B.A.; W. B. Grove, B.A.; *and others.*

The premises are (1) $axy+byz+czx : de+ef+fd$, and

(2) $d'x'y'+e'y'z'+f'z'x' : a'b'+b'c'+c'a'$.

By mere inspection, we get $xyz : a'b'c'+de+ef+fd$,

$x'y'z' : def+a'b'+b'c'+c'a'$, $x'yz : b'+de+ef+fd$, $xy'z : c'+de+ef+fd$,

$xyz' : a'+de+ef+fd$, $x'y'z : d+a'b'+b'c'+c'a'$, $xy'z' : e+a'b'+b'c'+c'a'$,

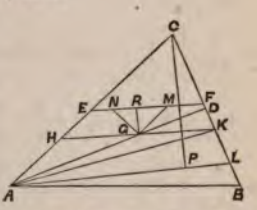
$x'yz' : f+a'b'+b'c'+c'a'$.

And the sum of all the antecedents implies the sum of all the consequents; that is, omitting redundant terms, $1 : a'+b'+c'+d+e+f$, which is equivalent to $abc : d+e+f$. This may be read, "If the events a, b, c all three happen, then one at least of the events d, e, f will happen.

5388 & 6471. (By Prof. Crofton, F.R.S.)—A triangle Δ is divided into two portions by a line joining two points taken at random within it; prove that the mean value (1) of the triangular portion is $\frac{4}{9}\Delta$; (2) of the portion that contains the centroid of the triangle is $\frac{\Delta}{3^6}(470+\frac{8}{3}^2\log 4)$; and (3) of the triangle formed by joining the centroid and the two points is $\frac{\Delta}{3^6}(34+\frac{1}{3}^6\log 2)$.

Solution by Professor Seitz, M.A.

Let ABC be the triangle, D the middle point of BC, G the centroid of the triangle, M and N two points such that EF, the line through them, meets AC and BC, and makes an angle with AC less than the angle A. Draw HGK and AL parallel to EF, and CP, GR perpendicular to AL, EF; and join AK. Let CP = x, EM = y, MN = z, EF = y', area CAL = u, CEF = v, CHK = v', and mean value throughout = M.



1. We have $AL = 2ux^{-1}$, and $y' = 2u^{\frac{1}{2}}v^{\frac{1}{2}}x^{-1}$. An element of surface at M is $\frac{1}{2}xu^{-\frac{1}{2}}v^{-\frac{1}{2}}dv\,dy$, and at N it is $\frac{1}{2}x^2u^{-2}du\,z\,dz$. The limits of u are 0 and Δ, of v, 0 and u; of y, 0 and y'; and of z, 0 and y, and doubled. Hence, since the whole number of ways the points can be taken is Δ^2,

$$\frac{2}{\Delta^2}\int_0^\Delta\int_0^u\int_0^{y'}\int_0^y \frac{x^3}{4x^{\frac{1}{2}}}\,du\,v^{\frac{1}{2}}\,dv\,dy\,z\,dz = \frac{2}{3\Delta^2}\int_0^\Delta\int_0^u \frac{du}{u}\,v^2\,dv = \frac{2}{9\Delta^2}\int_0^\Delta u^2\,du = \tfrac{2}{27}\Delta.$$

This result is the same as that for each of five other expressions similar to that above. Hence the mean value M required is $\frac{4}{9}\Delta$.

2. We have area $AKL = \frac{2}{3}ADL = \frac{2}{3}(u-\frac{1}{2}\Delta)$,

$ACK = u-\frac{2}{3}(u-\frac{1}{2}\Delta) = \frac{1}{3}(u+\Delta)$; $v' : \frac{1}{3}(u+\Delta) = \frac{1}{3}(u+\Delta) : u$, whence $v' = \frac{1}{9}(1+\Delta u^{-1})^2u$.

When $u < \frac{1}{2}\Delta$, the portion ABFE contains the centroid for all values of v; when $u > \frac{1}{2}\Delta$, the portion ABFE contains the centroid from $v = 0$ to $v = v'$, and the portion CEF contains it from $v = v'$ to $v = u$. Hence

$$\frac{2}{\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}\int_0^{y'}\int_0^{y} \tfrac{1}{4}x^3(\Delta - v)\,u^{-\frac{5}{2}}\,du\,v^{-\frac{1}{2}}\,dv\,dy\,z\,dz$$

$$+\frac{2}{\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(\Delta - v)\,v^{-\frac{1}{2}}\,dv + \int_{v'}^{u} v^{\frac{1}{2}}\,dv\right\}\int_0^{y'}\int_0^{y}\tfrac{1}{4}x^3 u^{-\frac{5}{2}}\,du\,dy\,z\,dz$$

$$= \frac{2}{3\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}(\Delta - v)\,u^{-1}\,du\,v\,dv + \frac{2}{3\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(\Delta - v)\,v\,dv + \int_{v'}^{u} v^2\,dv\right\}u^{-1}\,du$$

$$= \frac{1}{9\Delta^2}\int_0^{\frac{1}{2}\Delta}(3\Delta - 2u)\,u\,du + \frac{1}{9\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{2u + \frac{\Delta}{3^3}\left(1+\frac{\Delta}{u}\right)^4 - \frac{4u}{3^6}\left(1+\frac{\Delta}{u}\right)^6\right\}u\,du$$

$$= \frac{\Delta}{6\,.\,3^6}\left(470 + \frac{82}{3}\log 4\right);$$

hence $$M = \frac{\Delta}{3^6}\left(470 + \frac{82}{3}\log 4\right).$$

3. We have $GR = x^{-\frac{1}{2}}(v'^{\frac{1}{2}} - v^{\frac{1}{2}})$, or $xu^{-\frac{1}{2}}(v^{\frac{1}{2}} - v'^{\frac{1}{2}})$. When $u < \frac{1}{2}\Delta$, the area of the triangle MNG is $\frac{1}{2}xzu^{-}\ (v'^{\frac{1}{2}} - v^{\frac{1}{2}})$ for all values of v; when $u > \frac{1}{2}\Delta$, the area is the same as above from $v = 0$ to $v = v'$; but from $v = v'$ to $v = u$, the area is $\frac{1}{2}xzu^{-\frac{1}{2}}(v^{\frac{1}{2}} - v'^{\frac{1}{2}})$. Hence we have

$$\frac{2}{\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}\int_0^{y'}\int_0^{y}\tfrac{1}{8}x^4(v'^{\frac{1}{2}} - v^{\frac{1}{2}})\,u^{-3}\,du\,v^{-\frac{1}{2}}\,dv\,dy\,z^2\,dz$$

$$+\frac{2}{\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(v'^{\frac{1}{2}} - v^{\frac{1}{2}})\,v^{-\frac{1}{2}}\,dv + \int_{v'}^{u}(v^{\frac{1}{2}} - v'^{\frac{1}{2}})\,v^{-\frac{1}{2}}\,dv\right\}\int_0^{y'}\int_0^{y}\tfrac{1}{8}x^4 u^{-3}\,du\,dy\,z^2\,dz$$

$$= \frac{1}{9\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}(1 + \Delta u^{-1} - 3u^{-\frac{1}{2}}v^{\frac{1}{2}})\,u^{-\frac{1}{2}}du\,v^{\frac{3}{2}}\,dv$$

$$+\frac{1}{9\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}\left(1 + \frac{\Delta}{u} - \frac{3}{u^{\frac{1}{2}}}v^{\frac{1}{2}}\right)v^{\frac{3}{2}}\,dv + \int_{v'}^{u}\left(\frac{3}{u^{\frac{1}{2}}}v^{\frac{1}{2}} - 1 - \frac{\Delta}{u}\right)v^{\frac{3}{2}}\,dv\right\}u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{45\Delta^2}\int_0^{\frac{1}{2}\Delta}(2\Delta - 3u)\,u\,du + \frac{1}{45\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\frac{2}{3^6}(1 + \Delta u^{-1})^6 + 3 - 2\Delta u^{-1}\right\}u^2\,du$$

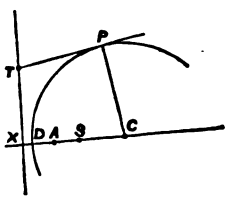
$$= \frac{\Delta}{6\,.\,3^6}\left(34 + \frac{16}{3}\log 2\right);$$

hence we have $$M = \frac{\Delta}{3^6}\left(34 + \frac{16}{3}\log 2\right).$$

6509. (By Prof. GENESE, M.A.)—Prove (1) that the directrices of a conic are common chords of the conic and its director circle; (2) that they are also the radical axes of that circle and of point-circles at the foci.

Solution by CHARLOTTE A. SCOTT; BELLE EASTON; *and others.*

Let A, S, C, XT be vertex, focus, centre and directrix of the conic; and CD = radius of director circle, so that $CD^2 = AC^2 + BC^2$; then, taking any point T on the directrix, drawing TP tangent to director circle, and joining TS, we have



$$TP^2 = TC^2 - CP^2 = TX^2 + XC^2 - AC^2 - CB^2$$
$$= TX^2 + XS^2 + SC^2 + 2SX.SC - AC^2 - BC^2$$
$$= TS^2 + 2SX.CS - 2BC^2.$$

Now we have $\frac{BC^2}{AC} : SX = SA : AX = CS : CA$;

hence $SX.CS = BC^2$, therefore $TP = TS$;

that is, T is on the radical axis of the director circle and a point circle at S; hence the directrix is this radical axis.

Let the ellipse and its director circle intersect at P, and draw PN perpendicular to the axis major, so that PN is common chord; then, since P is on the circle of centre C and radius CD, and also on conic, we have

$$PN^2 = CD^2 - CN^2, \quad PN^2 = \frac{BC^2}{AC^2}(AC^2 - CN^2) = BC^2 - \frac{BC^2}{AC^2}CN^2,$$

therefore $CD^2 - CN^2 = BC^2 - \frac{BC^2}{AC^2}.CN^2$;

but $CD^2 = AC^2 + BC^2$, therefore $CN^2 = \frac{AC^4}{AC^2 - BC^2} = \frac{CA^4}{CS^2}$.

But $CX^2 : AC^2 = AC^2 : CS^2$, hence $CX^2 = \frac{CA^4}{CS^2}$, $\therefore$ $CN = CX$;

therefore the directrix is the common chord.

6504. (By Prof. TOWNSEND, F.R.S.)—A system of smooth equal spheres, two terminal and fixed, and the remainder in contact each with the two adjacent, being supposed to form an arch in a vertical plane, in unstable equilibrium under the action of gravity; determine, given all particulars, the ultimate form of the arch, when the component spheres are indefinitely increased in number and diminished in magnitude.

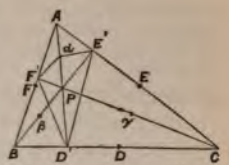
Solution by the Rev. A. L. WATHERSTON, M.A.; *the* PROPOSER; *and others.*

Every arc *s* of the curve of equilibrium of the system of spheres containing ultimately a number of spheres, and having, consequently, a mass proportional to its length, the s veral conditions for the equilibrium [unstable] of the system under the action of any forces are manifestly the same as for the equilibrium [stable] of a uniform flexible chord of the same length, mass, and terminal points, under the action of the same forces reversed in direction; and therefore, &c.

PROOF OF A NINE-POINT CIRCLE THEOREM. *By* C. PENDLEBURY, M.A.

Let the circle round D′E′F′ cut PA at α; then

$\angle$ F′αE′ = $\pi - \angle$ F′D′E′ = $\pi - (\pi - 2A) = 2A$.



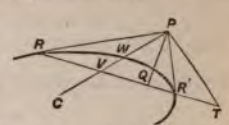
But a circle will go round AF′PE′, and PA is its diameter; hence α is its centre; therefore α, β, γ are the mid-points of PA, PB, PC.

[Mr. PENDLEBURY gives this proof as "much simpler and easier for boys to remember" than the proofs given in the usual text-books.]

5002. (By J. J. WALKER, M.A.)—Let S be a central conic, K its director circle, and S′ the reciprocal polar of K with respect to S; if Q is the pole of the tangent to K at P, prove that PQ is normal to S′; and that the normal to S, at the point where it is met by the connector of P with the centre, is parallel to PQ.

Solution by CHRISTINE LADD; J. O'REGAN; *and others.*

Draw PR, PR′ tangents from P to the conic S, and join CP, meeting RR′ in V, and the conic in W. Then RR′ is the tangent to the conic S′ at Q; also, since RPR′ is a right angle, VP = VR = VR′; and if PT, the tangent to the director circle, be drawn meeting RR′ produced in T Q is the pole of PQ; therefore RR′ is harmonically divided at Q and T; hence $VP^2 = VR'^2 = VQ \cdot VT$; therefore VQP, VPT are similar triangles, and VQP is a right angle. Hence PQ is normal to S′ at Q; also, the tangent at W being obviously parallel to RR′, the normal at W must be parallel to PQ.



SOLUTION OF A CUBIC EQUATION. *By* E. W. SYMONS, M.A.

Let the cubic be $x^3 + 3ax + b = 0$.....(1);

then, assuming $x = y^{\frac{1}{3}} + z^{\frac{1}{3}}$, we have $x^3 \equiv y + z + 3(yz)^{\frac{1}{3}} x$.....(2).

Making (1) and (2) identical, we have

$$y + z = -b, \quad yz = -a^3;$$

hence y and z are roots of the quadratic $u^2 + bu - a^3 = 0$;

therefore $$x = \left(\frac{-b + \sqrt{b^2 + 4a^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-b - \sqrt{b^2 + 4a^3}}{2}\right).$$

6560. (By W. B. Grove, B.A.)—In a certain school, every one who learns both English and French, or neither of them, does not learn Arithmetic; every one who learns Arithmetic learns both English and German or neither of them; everyone who learns French but not German learns either English or not Arithmetic. Assuming all combinations not inconsistent with these data to exist, prove (1) that the number of those who learn Arithmetic does not exceed the number of those who learn English; and (2) that the above conditions may be replaced by a single simple but exactly equivalent one.

Solution by H. McColl, B.A.; Elizabeth Blackwood; *and others.*

Speaking of some *one* boy, let a, e, f, g respectively denote the statements—he learns *Arithmetic*, he learns *English*, he learns *French*, he learns *German*. The data will then be expressed by the complex statement

$$(ef+e'f' : a')\,(a : eg+e'g')\,(fg' : e+a').$$

By inspection, we may reduce this to the form

$$a\,(ef+e'f'+e'g+eg'+e'fg') : 0.$$

Reducing the bracket factor in this implication to its *primitive form* (see my third paper in the *Proceedings of the Lond. Math. Soc.*), we get $a\,(e'+f+g') : 0$, that is, $a : ef'g$, as the simplest symbolical expression for the given conditions.

The number of boys to which the antecedent statement a is applicable is evidently an inferior limit to the number to which the consequent $ef'g$ is applicable, and, *a fortiori*, to the number to which e is applicable; that is to say, the number of boys that learn Arithmetic *cannot exceed* the number that learn English.

Note on the Solution of the 15-Puzzle in Question 6489 (*Reprint* Vol. XXXIV., p. 113).

By Rev. T. P. Kirkman, M.A., F.R.S.

My design was not simply to communicate the *rule* spoken of, which is well known, and has been given by Professor Tait in the Edinburgh Royal Society's *Proceedings*.

It appeared worth the while to show the connexion of the problem with the theory of substitutions. By whatever movement of the King S is reduced to S^0 = unity, as the natural order is called, the operation $S^{-1}S = 1$ has been performed by the unusual method of transpositions of an element undisturbed in S. In substitutions, $\phi\theta = \theta\phi$ is not generally true. It is true of $S^{-1}S = SS^{-1}$, because the product is a number. The results of the 13 operations (not upon S) indicated are

123856794	123476859	123875946
123896754	123479856	123895746
123876954	123475896	123845796
123976854	123975846	123849756

and $123846759 = S^2 = S^{3-1} = S^{-1}$.

The first, third, &c. are positive, none having an odd number of even circular factors. The second, fourth, &c. have each an odd number, and are negative. The second has one circle of 4, the fourth one of 2, the eighth one of 6. The terms positive and negative refer to the signs in the expansion of a determinant.

Unity being $123 \ldots 0a \ldots f$ of 16 elements,

$$769b310ca8e4d25f \text{ and } 314690ce8ba572df$$

are soluble, having for even circles 8, 2 and 12, 2,

$$234567890abcd1ef \text{ and } 2345619a0ec8bd7f$$

are insoluble, having one, the even circle 14 only, and the other 6, 4, 4. S has none, *i.e.*, an even number of even circles.

[On pp 58—68 and 187—192 of Vol. I. of the *Messenger of Mathematics* there are some hints on substitutions, by Mr. KIRKMAN.]

6516. (By C. W. MERRIFIELD, F.R.S.) — Prove that the remainders obtained by "casting out the nines" from the terms of any geometrical expression recur in regular order.

Solution by W. B. GROVE, B.A.; Prof. MATZ, M.A.; *and others.*

Let $a + ar + ar^2 + \ldots$ be the geometrical series, $a = 9m + n$, and $r = 9p + q$;

then $a + ar + ar^2 + \ldots = (9m + n) + (9m' + qn) + (9m'' + q^2 n) + \ldots$

Therefore the remainders are $n(1 + q + q^2 + \ldots)$; that is, the property in question will be true for all geometrical series, if it is true for all those which have the first term unity, and the common ratio any integer less than 9. On trial, this is found to be so for 1, 2, 4, 5, 7, 8, but not for 3, 6, or 9.

Therefore the theorem is true for all series in which the common ratio is not 3, 6, or 9.

6482. (By the EDITOR.)—Within a tetrahedron ABCD a point O is taken, and straight lines AOE, BOF, COG, DOH are drawn to meet the faces in E, F, G, H; find the average volumes of the tetrahedron EFGH.

Solution by Professor SEITZ, M.A.

Let vol. OBCD = w, vol. OCDA = x, vol. ODAB = y, vol. OABC = z, vol. ABCD = 1; then we have $w + x + y + z = 1$, and

vol. OEFG : vol. OABC = OE . OF . OG : OA . OB . OC ; whence

$$\text{vol. OEFG} = \frac{wxyz}{(1-w)(1-x)(1-y)} = \frac{wx}{(1-w)(1-x)}\left\{w+x+y-\frac{w+x}{1-y}\right\} = \text{V}.$$

The limits of w are 0 and 1; of x, 0 and $1-w$; and of y, 0 and $1-w-x$; hence the average volume of the tetrahedron OEFG is

$$\int_0^1\int_0^{1-w}\int_0^{1-w-x} \text{V}\,dw\,dx\,dy + \int_0^1\int_0^{1-w}\int_0^{1-w-x} dw\,dx\,dy$$

$$= 6\int_0^1\int_0^{1-w}\int_0^{1-w-x} \frac{wx}{(1-w)(1-x)}\left\{w+x+y-\frac{w+x}{1-y}\right\} dw\,dx\,dy$$

$$= 3\int_0^1\int_0^{1-w} \frac{w}{1-w}\left\{2w+x+(w+x)^2-\frac{w(2+w)}{1-x} - 2\left(1+w+x-\frac{1+w}{1-x}\right)\log(w+x)\right\} dw\,dx$$

$$= \int_0^1 \left\{10w+7w^2+w^3+\frac{6w^2(2+w)}{1-w}\log w - \frac{6w(1+w)}{1-w}\log w\log(1+w) - \frac{6w(1+w)}{1-w}\left[\frac{1}{1+w}+\frac{1}{2^2}\cdot\frac{1}{(1+w)^2}+\frac{1}{3^2}\cdot\frac{1}{(1+w)^3}+\ldots\right] + \frac{6w(1+w)}{1-w}\left[\frac{w}{1+w}+\frac{1}{2^2}\cdot\frac{w^2}{(1+w)^2}+\frac{1}{3^2}\cdot\frac{w^3}{(1+w)^3}+\ldots\right]\right\} dw$$

$$= 50\tfrac{1}{4} - 4\tfrac{1}{2}\pi^2 - 12\int_0^1 \left\{\frac{1}{1+w}\log w\log(1-w) + \frac{1}{1-w}\log w\log(1+w) - \frac{1}{w}\log(1-w)\log(1+w)\right\} dw$$

$$= 50\tfrac{1}{4} - 4\tfrac{1}{2}\pi^2 + 12\left\{\log w\log(1-w)\log(1+w)\right\}_0^1 - 24\int_0^1 \frac{dw}{1+w}\log w\log(1-w)$$

$$= 50\tfrac{1}{4} - 4\tfrac{1}{2}\pi^2 + 24\int_0^1 \left\{w-(1-\tfrac{1}{2})w^2+(1-\tfrac{1}{2}+\tfrac{1}{3})w^3 - (1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})w^4+\ldots\right\}\log w\,dw$$

$$= 50\tfrac{1}{4} - 4\tfrac{1}{2}\pi^2 - 24\text{S},$$

where $\text{S} \equiv \left\{\frac{1}{2^2}-(1-\frac{1}{2})\frac{1}{3^2}+(1-\frac{1}{2}+\frac{1}{3})\frac{1}{4^2}-(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4})\frac{1}{5^2}+\ldots\right\}$.

The average volume of each of the tetrahedrons OFGH, OGHE, OHEF is the same as that of OEFG; therefore the average volume of the tetrahedron EFGH is $201-18\pi^2-96\text{S}$; and as the value of the series S, taken to 50 terms, is ·243113, the average volume required is ·0084. [The analogous average for a *triangle* has (*Reprint*, Vol. IX., pp. 49—52, Quest. 2537) been shown to be $10-\pi^2$, or ·1304. The evaluation, in definite terms, if possible, of the series S, or of the integral that leads to it, has been proposed as Question 6608.]

6532. (By Professor SYLVESTER, F.R.S.)—Prove that the condition of an equation of the 5th degree being linearly transformable into a recurring form is $I=0$, where, $\alpha, \beta, \gamma, \delta, \epsilon$ being the 5 roots, I is the product of the 15 distinct factors obtainable from the permutations of

$$\pm[(\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma)+(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)],$$

and show that I is a symmetrical function of the 18th order in the coefficients of the given equation.

[The question may be otherwise stated as follows:—From the system

$$\frac{pm+q}{rm+s}=\alpha,\quad \frac{p\mu+q}{r\mu+s}=\gamma,\quad \frac{p+qm}{r+sm}=\beta,\quad \frac{p+q\mu}{r+s\mu}=\delta,\quad \frac{p+q}{r+s}=\epsilon,$$

eliminate p, q, r, s, m, μ, so as to obtain the relation between $\alpha, \beta, \gamma, \delta, \epsilon$ in an integral form.]

Solution by W. J. C. SHARP, M.A.

The relations given in the note lead to

$$p^2-q^2-(pr-qs)(\alpha+\beta)+(r^2-s^2)\alpha\beta=0,$$
$$p^2-q^2-(pr-qs)(\gamma+\delta)+(r^2-s^2)\gamma\delta=0,$$
$$p^2-q^2-2(pr-qs)\epsilon+(r^2-s^2)\epsilon^2=0,$$

which involve $(\epsilon-\beta)(\epsilon-\delta)(\gamma-\alpha)+(\epsilon-\alpha)(\epsilon-\gamma)(\delta-\beta)=0$, the relation given in the question; consequently the condition required will be the product of the fifteen distinct factors of this form which can be formed. It will therefore be of order $18(3\times2+12)$, and weight $4\cdot5$, and being a function of the differences will satisfy the known differential equation; it is in fact, as stated in the question, the invariant I mentioned in SALMON'S *Higher Algebra*, p. 189, and which Dr. SALMON has calculated. (*Phil. Trans.* for 1858, p. 455.)

[That this is correct follows from the fact that, if $(a.b\ldots f)(x,1)^5=0$ be a reciprocal equation, it has one root (±1) in common with its Canonizant, and therefore the eliminant I vanishes.]

6549. (By the Rev. W. A. WHITWORTH, M.A.)—If $(3N)!$ be divided by ϵ (the base of Napierian logarithms), prove that the integer nearest to the quotient will be a multiple of $3N-1$.

Solution by W. H. BLYTHE, B.A.; J. H. TURRELL, M.A.; *and others.*

$$e^{-1}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\&c.+(-1)^N\left\{\frac{1}{(3N-2)!}+\frac{1}{(3N-1)!}+\frac{1}{(3N)!}-\&c.\right\}$$

$$\therefore\ (3N)!\div e=3.4\ldots(3N-1).3N-4.5\ldots(3N-1).3N+\ldots$$
$$\ldots(-1)^N\left\{3N(3N-1)-3N+1-\frac{1}{3N+1}+\&c.\right\}$$

$$=\text{a multiple of } (3N-1)+(-1)^N\left\{-\frac{1}{3N+1}+\frac{1}{(3N+1)(3N+2)}-\&c.\right\}.$$

Now this latter series is negative and less than $\frac{1}{3N+1}$, that is, less than $\frac{1}{3}$; thus the nearest integer to (3N) ! + e is a multiple of (3N − 1).

[Mr. BLYTHE remarks that it appears to him that N ! + e is a multiple of (N − 1) ± a small fraction, according as N is odd or even, provided N is positive. To this Mr. WHITWORTH assents, and adds that if, moveover, f denote a small fraction, we have N ! + $e \pm f$ = M (N − 1), where M is an integer such that M ∓ 1 is a multiple of N (N − 3), the upper signs being taken when N is even and the lower when N is odd.]

6546. (By Professor PURSER, M.A.)—If the equation of a surface be expressed in the form $p = F(\theta, \phi)$, where p is the perpendicular from a fixed point on the tangent plane, θ, ϕ the usual angles determining its position; show that the sum of the principal radii of curvature is given by $R + R' = \Delta . p$, where Δ represents the differential operator

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2}{d\phi^2} + 2,$$

corresponding to the equation of LAPLACE's functions of Order 1. Hence, employing BOOLE's form of solutions, it appears that the general equations of surfaces having their principal curvatures equal and opposite is

$$p = \frac{d}{d\theta}[\sin\theta F_1(e^{i\phi}\tan\tfrac{1}{2}\theta) + \sin\theta F_2(e^{-i\phi}\tan\tfrac{1}{2}\theta)].$$

Solution by J. LARMOR, M.A.; Prof. SCHEFFER, M.A.; *and others.*

Consider the directions of the two lines of curvature through the point. The radius of curvature of the section through each is obviously the same as that of a cross-section of the cylinder formed by the tangent planes along it, and is therefore $p + \frac{d^2p}{d\phi^2}$, with the usual notation. Thus the sum of the principal radii is $\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2} + 2\right) p$, where ϕ, ψ are the angles p makes with two fixed planes at right angles.

Now consider a unit sphere round the origin. The *flux* of a function V through an element of area bounded by the small circles ϕ, $\phi + d\phi$, ψ, $\psi + d\psi$, is, as usual,

$$\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2}\right) V . d\phi\, d\psi,$$

therefore through any infinitesimal area is $\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2}\right) V$. area. Now, take a polar element of area, and take θ, ϕ as polar coordinates; the flux through this area is

$$\frac{d}{d\phi}\left(\frac{dV}{\sin\theta . d\phi} . d\theta\right) d\phi + \frac{d}{d\theta}\left(\sin\theta . d\phi . \frac{dV}{d\theta}\right) d\theta,$$

which is same as $\left\{\frac{1}{1-\mu^2}\frac{d^2V}{d\phi^2} + \frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu}\right\}$. area, where $\mu = \cos\theta$.

Thus the result follows, and the solution as deduced by Donkin in the *Phil. Trans.* for 1857. [Mr. Larmor remarks that the Cartesian equation of the tangent plane may now be written down, and its envelop found; the envelop being the most general solution of the equation to a soap-film, (Monge) $(1+x^2)\,r+2xys+(1+y^2)\,t=0$. In fact, if r^{-1} be put for p in Mr. Purser's solution, we obtain the polar equation of the general solution of the equation derived from this by "Reciprocal Transformation" (corresponding to Duality, or Reciprocal Polars in Geometry), that equation being $(1+y^2)\,r-2xys+(1+x^2)\,t=0$.]

4350. (By Colonel Clarke, C.B., F.R.S.)—Three points are taken at random in a given triangle; prove that the probability (p) that they enclose a given point whose triangular coordinates are (a, b, c), is

$$p = \mathrm{F}(a, b, c) + \mathrm{F}(b, c, a) + \mathrm{F}(c, a, b),$$

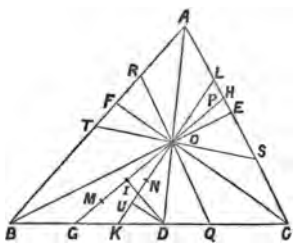
where $\mathrm{F}(a, b, c) = 2abc\,(1-6a^2) + (6b^2c^2 - 24b^3c^3)\log\left(1+\frac{a}{bc}\right)$.

5234. (By S. Watson.)—Three points are taken at random within a given triangle; prove that the chance that they will all lie on one side of some one line that can be drawn through the centroid of the triangle is

$$\frac{25}{27} - \frac{20}{81}\log 2.$$

Solution by Professor Seitz, M.A.

(4350.) Let ABC be the given triangle, O the given point, and M, N, P the three random points. Produce AO, BO, CO to D, E, F. Let M be confined to one of the six triangles into which the given triangle is divided, as BOD; the results for the other triangles will be similar to the result thus obtained. Let N be taken, successively, in the triangles BOD, COD, COE, AOE, AOF, BOF. Let KL be the line through N when it is taken in BOD or AOE, QR the line through it when taken in COD or AOF, and ST the line through it when taken in COE or BOF. Draw GH through M, and draw DI, DU perpendicular to GH, KL.



Let $DI = w$, DU or either of the perpendiculars from D, E, F on KL, QR, ST, $= x$, $OM = y$, $ON = z$, $OG = y'$, $OK = z'$, area $BOC = a$, area $COA = b$, area $AOB = c$, area $ABC = 1$, area $ODG = u$. Then we have

$$y' = 2uw^{-1},\quad \text{area ODB} = u_1 = \frac{ac}{1-a},\quad \text{area ODC} = u_2 = \frac{ab}{1-a},$$

$$\text{area OEC} = u_3 = \frac{ab}{1-b},\quad \text{area OEA} = u_4 = \frac{bc}{1-b},\quad \text{area OFA} = u_5 = \frac{bc}{1-c},$$

$$\text{area OFB} = u_6 = \frac{ac}{1-c},\quad \text{and area OEH} = u' = \frac{a^2b^2}{au_2+u} - bu_3;$$

and if v represents, successively, the areas ODK, ODQ, OES, OEL, OFR, OFT, we have area OEL $= v_1 = \dfrac{a^2b^2}{au_2+v} - bu_3$, area OFR $= v_2 = \dfrac{a^2c^2}{au_1+v} - cu_6$,

$$\text{area OFT} = v_3 = \frac{b^2c^2}{bu_4+v} - cu_5, \quad \text{area ODK} = v_4 = \frac{a^2b^2}{bu_3+v} - au_2,$$

$$\text{area ODQ} = v_5 = \frac{a^2c^2}{cu_6+v} - au_1, \quad \text{area OES} = v_6 = \frac{b^2c^2}{cu_5+v} - bu_4, \text{ and } z' = 2vx^{-1}.$$

An element of the triangle at M is $\frac{1}{2}w^2u^{-2}\,du\,y\,dy$, and at N it is $\frac{1}{2}x^2v^{-2}\,dv\,z\,dz$. When N is in the triangle ODB, the number of favourable positions of P is represented by the area of OHL; when N is in the triangle ODC, the number of favourable positions is represented by the area of OHAR; and so on. Hence, since the whole number of ways the three points can be taken is expressed by unity, we have

$$\begin{aligned}
&\int_0^{u_1}\Big\{\int_0^{u}(v_1-u')\,v^{-2}\,dv+\int_{u}^{u_1}(u'-v_1)\,v^{-2}\,dv+\int_0^{u_2}(u_4+u_5-u'-v_2)\,v^{-2}\,dv\\
&\quad+\int_0^{u_3}(u_4+u_5-u'+v_3)\,v^{-2}\,dv+\int_0^{u'}(1-u_2-u_3-u'-v_4)\,v^{-2}\,dv\\
&\quad+\int_{u'}^{u_4}(u_2+u_3+u'+v_4)\,v^{-2}\,dv+\int_0^{u_5}(u_2+u_3+u'-v_5)\,v^{-2}\,dv\\
&\quad+\int_0^{u_6}(u'+v_6)\,v^{-2}\,dv\Big\}\int_0^{y'}\int_0^{z'}\tfrac{1}{2}w^2x^2u^{-2}\,du\,y\,dy\,z\,dz\\
&=\int_0^{u_1}\Big\{\int_0^{u}(v_1-u')\,dv+\int_{u}^{u_1}(u'-v_1)\,dv+\int_0^{u_2}(u_4+u_5-u'-v_2)\,dv\\
&\quad+\int_0^{u_3}(u_4+u_5-u'+v_3)\,dv+\int_0^{u'}(1-u_2-u_3-u'-v_4)\,dv\\
&\quad+\int_{u'}^{u_4}(u_2+u_3+u'+v_4)\,dv+\int_0^{u_5}(u_2+u_3+u'-v_5)\,dv+\int_0^{u_6}(u'+v_6)\,dv\Big\}\,du\\
&=\int_0^{u_1}\Big\{2abc+4ab^2-6a^2b^2+\frac{2a^2b^2(1-4ab)}{au_2+u}-\frac{2a^4b^4}{(au_2+u)^2}+4a^2b^2\log\Big(1+\frac{u}{au_2}\Big)\\
&\qquad-2a^2b^2\log\Big(1+\frac{c}{ab}\Big)-2a^2c^2\log\Big(1+\frac{b}{ac}\Big)+2b^2c^2\log\Big(1+\frac{a}{bc}\Big)\Big\}\,du\\
&=2a^2bc+10a^2b^2c-2a^3b^3-8abcu_2+2a^2b^2u_3+2a^2b^2(1-4ab)\log\Big(1+\frac{c}{ab}\Big)\\
&\qquad+(4a^3b^2u_2+2a^2b^2u_1)\log\Big(1+\frac{c}{ab}\Big)-2a^2c^2u_1\log\Big(1+\frac{b}{ac}\Big)\\
&\qquad+2b^2c^2u_1\log\Big(1+\frac{a}{bc}\Big).
\end{aligned}$$

Writing out the results for the remaining five triangles, and adding together the six results, we find the probability to be as stated in the question. [Another Solution is given on p. 63 of Vol. XXI. of the *Reprint*.]

(5234.) The chance that the three points lie on one side of some one line that can be drawn through O is $p_1 = 1-p$. When O is the centroid of the triangle, $a = b = c = \frac{1}{3}$, and $p_1 = \frac{25}{27} - \frac{50}{81}\log 2$.

Generalization of Prevost and Lhulier's Theorem in Chances.

By E. B. Elliott, M.A.

Statement.—A bag contains a number, known or unknown, of balls, of which we are only told that each one must be of one or other of n different colours, there being no presumption *à priori* that any one possible combination of the colours is more likely than any other. Of these balls $\Sigma(p)$ are now drawn, and found to be $p_1, p_2, p_3, \ldots p_n$ of the n colours severally, and are not replaced. To show that, if $\Sigma(\varpi)$ more be drawn, the chance that they be $\varpi_1, \varpi_2, \ldots \varpi_n$ of the colours respectively, is

$$\frac{\{\Sigma(\varpi)\}\,!\,\Pi\{(p+\varpi)\,!\}\,\{\Sigma(p)+n-1\}\,!}{\Pi(\varpi\,!)\,\Pi(p\,!)\,\{\Sigma(p+\varpi)+n-1\}\,!}.$$

Prevost and Lhulier's theorem (Todhunter's *History of Probability*, p. 454) is the particular case of this for two colours. For simplicity of expression, the statement has been made as to balls taken from a bag, but it will be at once clear that the same result applies to many other classes of observations.

Proof.—We should not have affected either the observation or the chance if, before drawing a ball, we had put our hand in the bag and fixed upon the $\Sigma(p)+1$ balls destined to be taken out in the first $\Sigma(p)+1$ drawings. Now, all possible hypotheses as to the combination of colours in the whole number of balls being *à priori* equally likely, it follows that all hypotheses possible with regard to $\Sigma(p)+1$ balls taken at random from among them are also equally likely. But of these various hypotheses only n are still tenable when of the $\Sigma(p)+1$ balls all but one have been taken up and found to be $p_1, p_2, \ldots p_n$ of the several colours; viz., the hypotheses that there were, of those colours respectively,

$$p_1+1, p_2, p_3, \ldots p_n;\ p_1, p_2+1, p_3, \ldots p_n;\ \ldots;\ p_1, p_2, p_3, \ldots p_n+1 \ldots (1, 2, 3).$$

Now, the chance of the observed event, on the supposition in turn of each of these hypotheses, is just that in each case that the single ball left to the $\Sigma(p)+1^{\text{th}}$ drawing be of the singular colour 1, 2, 3, ... or n. That is to say, the several chances are proportional to

$$p_1+1,\ p_2+1,\ p_3+1, \ldots p_n+1,$$

the common denominator being $\Sigma(p)+1$.

Hence the chances of the truth of the various hypotheses, estimated after the observed event, have the same numerators as these chances, their common denominator being the sum of those numerators, *i.e.* $\Sigma(p)+n$.

These are the chances after the $\Sigma(p)$ observed drawings, that the next one will give a ball of the first, second, ... n^{th} colour respectively.

Now, suppose this one drawing to be made, and that the resulting colour prove to be the r^{th}. Then the chances for the various colours at the next drawing are in like manner

$$p_1+1, p_2+1, \ldots p_r+2, \ldots p_n+1,$$

with common denominator $\Sigma(p)+n+1$.

Thus, returning to the moment after the first $\Sigma(p)$ drawings, the chance then that the next drawing will give colour r, and the one following colour s, is $(p_r+1)(p_s+1) \div \{\Sigma(p)+n\}\{\Sigma(p)+n+1\}$, unless $s=r$, in which case it is

$$(p_r+1)(p_r+2) \div \{\Sigma(p)+n\}\{\Sigma(p)+n+1\}.$$

We see also from the general form that the chances for the orders r, s and s, r are the same.

We may proceed now in this way to the consideration of any number we please of future drawings after the observed $\Sigma(p)$, and find at once that the chance in favour of the first $\Sigma(\varpi)$ drawings subsequent giving $\varpi_1, \varpi_2, \ldots \varpi_n$ of the various colours in any assigned order that we choose to fix upon, is

$$\frac{(p_1+1)(p_1+2)\ldots(p_1+\varpi_1)\,.\,(p_2+1)\ldots(p_2+\varpi_2)\,.\,\ldots\,(p_n+1)\ldots(p_n+\varpi_n)}{\{\Sigma(p)+n\}\{\Sigma(p)+n+1\}\ldots\{\Sigma(p)+\Sigma(\varpi)+n-1\}},$$

a result, quite independent of that particular order, which we may write

$$\frac{(p_1+\varpi_1)!\,(p_2+\varpi_2)!\ldots(p_n+\varpi_n)!\,\{(\Sigma p)+n-1\}!}{p_1!\,p_2!\,\ldots\ldots p_n!\,\{\Sigma(p+\varpi)+n-1\}!},$$

or

$$\frac{\Pi\{(p+\varpi)!\}\{\Sigma(p)+n-1\}!}{\Pi(p!)\{\Sigma(p+\varpi)+n-1\}!}.$$

To find the chance of the $\Sigma(\varpi)$ drawings giving the required colour combinations in some order or other, we must multiply this by the number of different possible orders, *i.e.*, by $\{\Sigma(\varpi)\}!\div\Pi(\varpi!)$. And doing this we get the value given in statement.

Test of Accuracy. — Putting $p_1 = p_2 = \ldots = p_n = 0$, and letting $\varpi_1+\varpi_2+\ldots+\varpi_n = m$, the whole number of balls in the bag, substitution in one result gives for the chance in the first place of any particular division into parcels of the n colours of all the balls $1 \div$ the whole number of ways in which these balls can be divided into n parcels, a result independent of that particular division. And this was the supposition we started from.

Particular Example. — Referring to Quests. 4549, 4566, 4578, 4579 (*Reprint*, Vol. XXXIII., p. 68), I agree with Mr. TEBAY that there is no ambiguity in the questions, but submit that their only admissible meaning is the one which Mr. SANDERSON and myself attached to them, and not the one on which he and Mr. LEUDESDORF have based their solutions. It seems to me that the question which they have solved is this: "A bag contains mn balls, of which every one must be of one or other of n different colours, *all different possible combinations of those colours being, it is presumed, equally likely.* All but m are drawn at random, and found to be $n-1$ of each colour. Find &c. &c." Thus stated, the theorems are included in the above general one, and the forms of result written down by Mr. LEUDESDORF (without proof for the general case) are immediate by substitution.

5248. (By Professor WOLSTENHOLME, M.A.)—If O, A, B be three points in order on a straight line, and a point P move so that OP is always a mean proportional between AP, BP; prove that (1) there exists also another system of three points A′, O′, B′, in order on the same straight line, such that O′P is a mean proportional between A′P, B′P; (2) O, O′ divide AB and A′B harmonically, and

$$OA+OB : OA'+OB' = OA\,.\,OB : OA'.\,OB' = (OA+OB)^2 : AB^2;$$

(3) the four points A, B, A′, B′ are the axial foci of the curve, which is a circular cubic, and the circle on OO′ as diameter is one of the circles with

respect to which the curve is its own inverse passing through four impossible foci; and (4) the vector equations of the curve are, if $AP = r_1$, $BP = r_2$, $A'P = r_3$, $B'P = r_4$, and $4n = \dfrac{AB^2}{OA \cdot OB}$,

$$2(n-1)^{\frac{1}{2}} r_2 + [2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_3 + [-2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_4 = 0,$$
$$2(n-1)^{\frac{1}{2}} r_1 + [2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_3 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_4 = 0,$$
$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}] r_2 + 2(n+1)^{\frac{1}{2}} r_4 = 0,$$
$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}] r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}] r_2 + 2(n+1)^{\frac{1}{2}} r_3 = 0.$$

Solution by Prof. NASH, M.A.; J. HAMMOND, M.A.; *and others.*

Taking O as origin, $OA = a$, $OB = b$, $OO' = \alpha$, $OA' = \beta$, $OB' = \gamma$, we have to show that α, β, γ can be so determined that the equations (1) and (2) are identical, $(x^2+y^2)^2 = [(x-a)^2+y^2]\,[(x-b)^2+y^2]$ .....(1),

$$[(x-\alpha)^2+y^2]^2 = [(x-\beta)^2+y^2]\,[(x-\gamma)^2+y^2] \quad \text{.....(2).}$$

The necessary conditions are

$$\frac{\beta+\gamma-2\alpha}{a+b} = \frac{\beta^2+\gamma^2-2\alpha^2}{a^2+b^2} = \frac{\beta\gamma-\alpha^2}{ab} = \frac{\beta\gamma(\beta+\gamma)-2\alpha^3}{ab(a+b)} = \frac{\beta^2\gamma^2-\alpha^4}{a^2b^2} = k.$$

From these equations, we obtain

$$\beta+\gamma+2\alpha = a+b, \qquad \beta\gamma+\alpha^2 = ab,$$
$$\beta+\gamma-2\alpha = k(a+b), \qquad \beta\gamma-\alpha^2 = kab;$$

therefore
$$\beta+\gamma = \tfrac{1}{2}(1+k)(a+b), \qquad \beta\gamma = \tfrac{1}{2}(1+k)\,ab,$$
$$2\alpha = \tfrac{1}{2}(1-k)(a+b), \qquad \alpha^2 = \tfrac{1}{2}(1-k)\,ab;$$

therefore $\tfrac{1}{2}(1+k) = \left(\dfrac{a-b}{a+b}\right)^2 = \dfrac{n}{n+1}$, $\tfrac{1}{2}(1-k) = \dfrac{4ab}{(a+b)^2} - \dfrac{1}{n+1}$;

therefore
$$\frac{\beta+\gamma}{a+b} = \frac{\beta\gamma}{ab} = \frac{(a-b)^2}{(a+b)^2} = \frac{n}{n+1};$$

i.e.,
$$\frac{OA - OB}{OA' + OB'} = \frac{OA \cdot OB}{OA' \cdot OB'} = \frac{(OA+OB)^2}{AB^2}.$$

Also
$$\frac{2}{OO'} = \frac{2}{\alpha} = \frac{8}{(1-k)(a+b)} = \frac{a+b}{ab} = \frac{1}{a}+\frac{1}{b} = \frac{1}{OA}+\frac{1}{OB}$$
$$= \frac{\beta+\gamma}{\beta\gamma} = \frac{1}{OA'}+\frac{1}{OB'};$$

therefore OO′ divides AB and A′B′ harmonically. Therefore A′, O′, B are in order.

The form of (1) shows that $y \pm i(x-a) = 0$ are tangents, therefore A is a focus, and similarly A′, B, B′ are foci.

Taking the middle point of OO′ as origin, the equation of the locus is

$$[2x(a+b) - (a-b)^2](x^2+y^2) + \frac{2a^2b^2}{a+b}x = 0,$$

or
$$2r^2\cos\theta(a+b) - r(a-b)^2 + \frac{2a^2b^2}{a+b}\cos\theta = 0,$$

and this is evidently its own inverse with respect to the circle $r = \dfrac{ab}{a+b}$, *i.e.*, the circle on OO′ as diameter. It is proved, in the solution

of Quest. 5878, that if d_1, d_2, d_3, d_4 be the distances of the foci from the foot of the asymptote, the vector equations are of the form

$$(d_3^2-d_4^2)\,r_2+(d_4^2-d_2^2)\,r_3+(d_2^2-d_3^2)\,r_4=0.$$

In this case the asymptote is $x=\dfrac{a^2+b^2}{2(a+b)}=\dfrac{2n+1}{4(n+1)}(a+b)$, and the distances d_1, d_2, d_3, d_4 are proportional to

$$1\pm 2[n(n+1)]^{\frac{1}{2}},\quad -1\pm 2[n(n-1)]^{\frac{1}{2}}.$$

Hence the four vector equations can be found. These equations can also be found linearly by taking the distances of the vertices from the foci.

6551. (By R. A. Roberts, M.A.)—If three lines, making an angle θ with a conic at points whose eccentric angles are α, β, γ, meet in a point, show that $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=\dfrac{2ab}{a^2-b^2}\cot\theta$.

Solution by Prof. Wolstenholme, M.A.; E. Rutter; *and others.*

If OP, OQ, OR be the three lines, the triangle formed by other three straight lines drawn through P, Q, R at angles to OP, OQ, OR, will be determinate in form whatever ϕ may be, and any one of the sides will vary as $\sin\phi$. Hence, in the two triangles formed by the normals, and by the tangents, at P, Q, R, the ratio of homologous sides will be $\sin(\frac{1}{2}\pi-\theta)$: $\sin\theta$, or $\cot\theta$: 1. But this ratio is well known to be

$$\frac{a^2-b^2}{2ab}[\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)]:1.$$

Hence, if two concentric conics be such that triangles PQR can be inscribed in one whose sides touch the other, straight lines can be drawn through P, Q, R to a point O such that OP, OQ, OR are inclined to the respective normals at P, Q, R at the same angle which remains constant for all the triangles.

[If the triangles be respectively inscribed in and circumscribed to the ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $A\dfrac{x^2}{a^2}+B\dfrac{y^2}{b^2}+2H\dfrac{xy}{ab}=1$, the angle will be

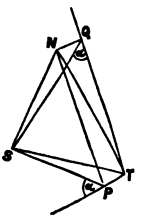
$$\tan^{-1}\left(\frac{a^2-b^2}{4ab}\cdot\frac{H}{H^2-AB}\right).$$

A, B, H are connected by the equation $(A+B+H^2-AB)^2=4(AB-H^2)$.]

6026. (By W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T.

Solution by D. EDWARDES; Prof. NASH, M.A.; *and others.*

By taking a consecutive position of the tangents, NT is seen to be the normal at T to the locus of T. Join SN, ST. A circle will evidently pass through SPTQN. Hence $\angle$ STN = $\angle$ SQN, and consequently the locus of T is a *similar* equiangular spiral. Again, $\angle$ SNT = $\angle$ SQT = the constant angle of spiral. But this is the construction for the centre of curvature of the equiangular spiral described by T. Hence the locus of N is the evolute of the locus of T.



6426. (By C. LEUDESDORF, M.A.)—Prove that, if n be odd, the product of the squares of the differences of the roots of the equation

$$x^n + nx^{n-1} + (n-1)^{n-1} = 0 \quad \text{is} \quad 2n^n (n-1)^{(n-1)^2}.$$

Solution by the Rev. D. THOMAS, M.A.; H. HILLARY, M.A.; *and others.*

Let $a_1, a_2, \ldots a_n$ be roots of $f(x) = x^n + nx^{n-1} + (n-1)^{n-1} = 0$; then

$$(a_1-a_2)^2 (a_1-a_3)^2 \ldots = (-1)^n f'(a_1) f'(a_2) \ldots f'(a_n)$$

$$= -[na_1^{n-1} + n(n-1) a_1^{n-2}] [na_2^{n-1} + n(n-1) a_2^{n-2}] \ldots (n \text{ being odd})$$

$$= -n^n (a_1 a_2 \ldots a_n)^{n-2} (n-1+a_1)(n-1+a_2)(n-1+a_3)\ldots$$

$$= -n^n (n-1)^{(n-1)(n-2)} f\{-(n-1)\}$$

$$= n^n (n-1)^{(n-2)(n-1)} \{-(n-1)^n + n(n-1)^{n-1} + (n-1)^{n-1}\}$$

$$= n^n (n-1)^{(n-1)^2} [-(n-1) + n + 1] = 2n^n (n-1)^{(n-1)^2}.$$

4795 & 4198. (By Prof. CROFTON, F.R.S.)—Prove that—(1) the arc of a Cartesian oval, at any point p, is equally inclined to the straight line from p through any one point, and to the circular arc from p through the other two foci; and (2), any system of triconfocal Cartesian ovals intersect each other orthogonally. [By triconfocal is meant having their three collinear foci common. Confocal conics furnish a particular case of this theorem, as they have a third common focus at an infinite distance.]

Solution by Professor NASH, M.A.

I assume the following theorem, which can be very easily proved:—If the constant of inversion be so chosen that a Cartesian is its own inverse with respect to one focus A, then the other two foci B, C are inverse

points; or, in another form, If a circle through BC meet the curve in P, and AP meet the circle again in P′, then P′ is also upon the Cartesian."

Now, taking two circles, centres A, B and radii a, b, and any point O in the line AB, if through O a line be drawn cutting the circles in Q, R and Q′, R′, the four intersections of the lines AQ, AR with the lines BQ′, BR′ all lie upon a Cartesian having A, B as foci.

Also, the tangent at any one of these points, and the corresponding tangents to the circles, are concurrent. (CHASLES's *Aperçu Historique*, p. 853.)

In the figure, considering OQQ′ as a transversal cutting the sides of the triangle PAB, we get at once the vector equation

$$\frac{OA}{a}.AP-\frac{OB}{b}.BP=AB.$$

Now, consider the four points in which AQ meets the Cartesian; two of these, P, P′, are determined as above by the line OQ, and the other two by the line Oq, Q$A$$q$ being a diameter of the circle. Hence P, P′ are inverse points with respect to A, therefore the circle through PP′B passes through the third focus C.

If PK be the tangent at P to this circle, the angle KPA is equal to the angle PBP′; also, the angles PQT, PQ′T are right angles, therefore the angle QPT = angle QQ′T = ½ angle PBP′, therefore the angle KPA = double the angle TPQ.

If the tangents at P, P′ meet in Y, the angles YPP′, YP′P are equal; therefore YP = YP′, and Y lies on the circle round BCPP′.

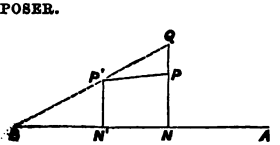
Hence also, if two Cartesians have the same foci, they will cut each other at right angles, which is Quest. 4198.

[For another solution of Quest. 4795, see *Reprint*, Vol. XXV., p. 17.]

6276. (By Prof. TANNER, M.A.)—In measuring the distance between two "stations," a surveyor, holding one end of the measuring chain at the near station, places his assistant, who has the other end of the chain, in a direct line between himself and the further station. The assistant leaves a mark from which the operation is repeated, and so on until the line is finished. Suppose that, instead of directing the assistant from the proper place, the surveyor always stands a small distance (h) to the right; find (1) the deviation produced after measuring n chains, the distance between the stations being m chains; and (2) the maximum deviation.

Solution by the PROPOSER.

In the figure, AB is the line to be measured, and P is the end of the n^{th} chain. PQ = h, PP′ = 1, PN = δ_n, P′N′ = δ_{n+1}. Also, very nearly, BN = $m-n$, BN′ = $m-n-1$ (all distances being in "chains").



From the figure, $$\frac{\delta_{n+1}}{m-n-1}=\frac{\delta_n+h}{m-n};$$

whence $$\delta_n=(m-n)\,h\left\{\frac{1}{m}+\frac{1}{m-1}+\ldots+\frac{1}{m-n+1}\right\}=m-n\,.\,h\,.\,S_n,\text{ say.}$$

Therefore $\delta_{n+1} - \delta = h\,(1 - S_{n+1})$.

Hence the necessary and sufficient conditions that δ_n is a maximum are $S_n < 1 < S_{n+1}$.

If $S_{n+1} = 1$, δ_n and δ_{n+1} have the same value, and each is a maximum.

[The determination of n from these data has been proposed for solution as a separate Question.]

6464. (By W. S. B. Woolhouse, F.R.A.S.)—Show that

$$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2 \cdot \frac{2 \cdot 4 \ldots 2n}{3 \cdot 5 \ldots 2n+1}\,(a+x)^{\frac{1}{2}} \left\{ a^n \left(1 + \frac{a}{x}\right)^{-\frac{1}{2}} \text{ as far as } x^n \right\}.$$

Solution by G. F. Walker, M.A.; R. Knowles, L.C.P.; *and others.*

$$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2x^n\,(a+x)^{\frac{1}{2}} - 2n \int \frac{x^{n-1}(a+x)\,dx}{(a+x)^{\frac{1}{2}}},$$

$$(2n+1) \int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2x^n\,(a+x)^{\frac{1}{2}} - 2na \int \frac{x^{n-1}\,dx}{(a+x)^{\frac{1}{2}}};$$

therefore $$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = \frac{2x^n\,(a+x)^{\frac{1}{2}}}{2n+1} - \frac{2na}{2n+1} \int \frac{x^{n-1}\,dx}{(a+x)^{\frac{1}{2}}} = \&c.;$$

and since coefficient of x^n in $\left(1 + \frac{x}{a}\right)^{-\frac{1}{2}}$ is $\frac{(-1)^n}{a^n} \frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2^n\,(n)!}$, the result follows.

6522. (By W. R. Westropp Roberts, M.A.)—A sphere is gently placed on a rough inclined plane of given height and length, and descends under the influence of gravity; prove that the value of the coefficient of friction for which the *vis viva* gained in the descent is a minimum is

$$2mgh \times \frac{2a}{a + (a^2 + k^2)^{\frac{1}{2}}}.$$

Solution by G. S. Carr, B.A.

Let the acceleration of the centre of the sphere be f, and the angular acceleration F; then, with the usual notation,

$$f = g\,(\sin\alpha - \mu\cos\alpha), \quad \mathrm{F} = \frac{a}{k^2}\,\mu g \cos\alpha,$$

both being constants. Therefore, t being the time of descent, $v = ft$, $\omega = \mathrm{F}t$, $v^2 = 2fh \operatorname{cosec}\alpha$. The *vis viva* acquired will be

$$\mathrm{V} = m\,(v^2 + k^2\omega^2) = mv^2 \left(1 + k^2 \frac{\mathrm{F}^2}{f^2}\right).$$

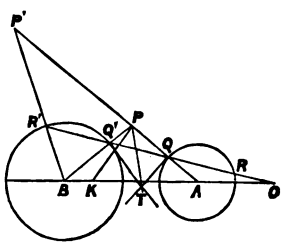
Substitute the values of F, f, v^2, and put $\mu = x \tan a$, $k^2 = \frac{2}{3}a^2$; then

$$V = mgh\frac{2-4x+7x^2}{1-x}.$$ Then $\frac{dV}{dx} = 0$ gives $x = 1 \pm \sqrt{\frac{2}{7}}$.

Since x must be $< \frac{1}{7}$, the least root is the one required. Substituting this value of x in V, we obtain the given result.

3930. (By Professor WOLSTENHOLME, M.A.) — In the limaçon $r = a(e+\cos\theta)$, if S be the pole, O the centre of inversion, OpP any chord through O, and y the point of intersection of the tangents at P, p, then y lies on the circle SpP, which touches the axis at S; Py = py, and the locus of y is the cissoid $r\cos\theta + a\sin^2\theta = 0$.

Solution by Prof. SCHEFFER, M.A.; Prof. NASH, M.A.; *and others.*



The first part of this is a particular case of the Corollary given in my solution of Quest. 4795, the two foci B, C coinciding at the pole of the limaçon, and the single focus A being the centre of inversion, or point, with respect to which the curve is its own inverse.

Inverting with respect to S, the proposition becomes this: Pp is a chord of a central conic parallel to the transverse axis, two circles pass through a focus S and touch the conic at P, p respectively; the second point (y) of intersection of these circles lies upon Pp; SP : Py = Sp : py; and the locus of y is a parabola having its vertex at S.

The point y lies on the line joining S to the pole of Pp; therefore Sy bisects $\angle$ PSp. The rest is easily proved.

6421. (By W. H. H. HUDSON, M A.) — If a point P moves so that SP . PM is constant, where S is a fixed point and PM is perpendicular to a fixed line, and if tangents at any two points of its path be drawn and produced to meet the asymptote, prove that the triangle included between the two tangents and the asymptote is equal to the area included between them and the corresponding arc of the locus of the extremity of the polar subtangent.

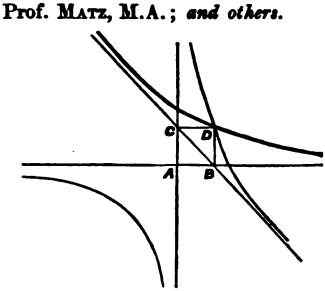
Solution by G. F. WALKER, M.A.; J. O'REGAN; *and others.*

It is a known property of this curve, that the part of the tangent intercepted between the extremity of the polar subtangent and the line which is the locus of M, is bisected at the point of contact.

Hence the whole area swept out by this line between the curve, tangents, and the line, will be equal to that between the curve, tangents, and the locus of the polar subtangent; and, since the part between the tangents and the curve will be common to both, the result follows.

5869. (By C. Leudesdorf, M.A.)—**ABDC is a square. Find the equation, referred to AB, AC as axes, of the cubic curve having AB, BC, CA as asymptotes, and having a double point at D: and trace the curve completely.**

Solution by Prof. Nash, M.A.; Prof. Matz, M.A.; *and others.*



The equation of a cubic having $x = 0$, $y = 0$, $x+y-a = 0$ for asymptotes, must be of the form

$$xy(x+y-a) + Ax + By + C = 0,$$

which, transformed to parallel axes through D, becomes

$$xy(x+y) + a(x^2+3xy+y^2) + x(A+2a^2) + y(B+2a^2) + Aa + Ba + C + a^3 = 0.$$

Since D is a double point, we have

$$A = -2a^2 = B,\ C = 3a^3,$$

and the equations referred to AB, AC and DB, DC are

$$xy(x+y-a) = a^2(2x+2y-3a),\quad xy(x+y) + a(x^2+3xy+y^2) = 0 \ldots (A,B).$$

From (B) we see that the approximate form of the curve at D is $y^2+3xy+x^2 = 0$, and the tangents to the two branches make angles $\tan^{-1}\frac{1}{2}(-3\pm\sqrt{5})$ with the axis of x. The curve cuts the asymptotes $y = 0$, $x = 0$ at the points $x = \frac{3}{2}a$, $y = 0$, and $x = 0$, $y = \frac{3}{2}a$, but does not meet the third asymptote in any point at a finite distance from A. From (A),

$$\frac{dy}{dx} = \frac{2a^2 - y(x+y-a) - xy}{x(x+y-a) + xy - 2a^2} = -\frac{8}{5} \quad \text{when } x = \tfrac{3}{2}a,\ y = 0.$$

The curve is evidently symmetrical with respect to the line $x = y$, and meets it again in the point $x = y = -\frac{3}{2}a$; therefore the form is that shown in the figure.

6539. (By Prof. Minchin, M.A.)—Prove the following fundamental theorem:—In all motion of a rigid body parallel to one plane, there is at every instant a point which has no acceleration (let us call it the *instantaneous acceleration centre*, and denote it by J); the acceleration of every point in the body is directly proportional to its distance from J, and its

direction makes a constant angle with the line joining the point to J. [If P is any point in the body, and ω, $\dot{\omega}$ the angular velocity and angular acceleration of the body at any instant, the acceleration of P is $(\omega^4+\dot{\omega}^2)^{\frac{1}{2}}$. JP, and $\tan\phi = \frac{\dot{\omega}}{\omega^2}$, where ϕ = angle between JP and direction of acceleration. Hence there is construction for J analogous to that for I, the instantaneous velocity centre.]

I. *Solution by* G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others.*

Let $\ddot{x}$, $\ddot{y}$, $\ddot{\theta}$ be the accelerations of the centre of gravity, and the angular acceleration about it, and let (x, y) be the coordinates of any point of the body relative to the centre of gravity. The acceleration of this point will be (since its distance from the centre of gravity is constant),

$$\ddot{x}-x\dot{\theta}^2-\ddot{\theta}y,\quad \ddot{y}-y\dot{\theta}^2+\ddot{\theta}x \qquad \ldots\ldots(1).$$

If we choose x_1, y_1 so that $\ddot{x}-x_1\dot{\theta}^2-\ddot{\theta}y_1 = 0$, and $\ddot{y}-y_1\dot{\theta}^2+\ddot{\theta}x_1 = 0$, we get the position of J, the acceleration centre, and the expressions (1) become $\dot{\theta}^2(x_1-x)+\ddot{\theta}(y_1-y)$ and $\ddot{\theta}(x_1-x)-\dot{\theta}^2(y_1-y)$. Squaring and adding, we find $(\dot{\theta}^4+\ddot{\theta}^2)^{\frac{1}{2}}$JP for the complete acceleration, and the form of the expressions shows that the direction makes an angle $\tan^{-1}\frac{\ddot{\theta}}{\dot{\theta}^2}$ with JP.

II. *Solution by* J. J. WALKER, M.A.

Let O, O′ be the centres of instantaneous angular velocity and acceleration respectively. Describe a circumference on OO′ as diameter, and in OO′ take Q, so that $OQ : QO' = \dot{\omega}^2 : \omega^4$. Draw J′QJ perpendicular to OO′ to meet the circumference in J′J, J being the one of the two points such that $\dot{\omega}$ is in the direction OJ. Then J will be the "instantaneous acceleration centre," since $OJ\omega^2 = O'J\dot{\omega}$ .....(1).

Let P be any other point in the same plane. Its accelerations will be $\dot{\omega}$. O′P, perpendicular to O′P, and ω^2OP in the direction PO. Resolving these in the direction PJ, and in that of a perpendicular to PJ, the components of their resultant (a) in these directions will be

$$X = \omega^2 OP\cos OPJ - \dot{\omega}O'P\sin O'PJ = \omega^2(JP + OJ\sin O'JP) - \dot{\omega}O'P\sin O'PJ$$
$$= \omega^2 JP \text{ in virtue of (1)}; \text{ and}$$

$$V = \omega^2 OP\sin OPJ + \dot{\omega}O'P\cos O'PJ = \omega^2 OJ\cos O'JP + \dot{\omega}O'J\cos O'PJ\sin O'JP$$
$$+ \sin O'PJ = \dot{\omega}O'J\sin PO'J + \sin O'PJ \text{ by (1)} = \dot{\omega}.JP.$$

Hence, at once, $a = (\omega^4+\dot{\omega}^2)^{\frac{1}{2}}$JP, $\tan\phi = \dot{\omega}\div\omega^2$.

[Mr. WALKER states that the foregoing solution was written out before he had seen Prof. EVERETT's reference to SCHELL's *Memoir* (1873) as having anticipated Prof. MINCHIN in the discovery of the "Centre of Acceleration." To SCHELL's acknowledgment of the priority of BRESSE (1853) in noticing the point, and its properties, as given in the Question, Mr. WALKER had elsewhere called attention. BRESSE's method of treating the Question is, however, quite different from the above, which is more like SCHELL's, but less diffuse.]

6405 & 6531. (By Prof. SYLVESTER, F.R.S.)—If p, q, r, s are the distances of a point in a circular cubic from the four concyclic foci A, B, C, D; prove (1) that

$$\frac{(p-q)(q-r)(r-p)}{\text{ABC}} = \frac{(q-r)(r-s)(s-q)}{\text{BCD}} = \frac{(r-s)(s-p)(p-r)}{\text{CDA}} = \&c.;$$

also prove (2) that only one proper circular cubic can be drawn having four concyclic foci at the angles of a trapezoid.

I. *Solution by* W. R. WESTROPP ROBERTS, M.A.

Let O be a centre of the quadrangle formed by A, B, C, D, the four concyclic foci of the cubic, and let the distances of O from the latter points be a, b, c, d respectively; then we have

$$\begin{Vmatrix} p & q & r & s \\ a & b & c & d \\ 1 & 1 & 1 & 1 \end{Vmatrix} = 0, \quad \therefore \frac{(p-q)(q-r)(r-p)}{(a-b)(b-c)(c-a)} = \frac{(s-p)(p-q)(q-s)}{(d-a)(a-b)(b-d)} = \&c.$$

Now, it is easy to see that $\Sigma'_1 p^2(b-c)(c-d)(d-a) = 0$; but since p^2, q^2, r^2, s^2 are point circles, we have also $\Sigma p^2 \text{BCD} = 0$, hence $(d-a)(b-c)(c-d)$ is proportional to BCD; therefore, &c.

II. *Solution by* W. J. C. SHARP, M.A.

The equation to any circular cubic can be reduced to the form

$$x(x^2+y^2) = ax^2 + 2bxy + 2fy + 2gx,$$

equivalent to $(x+\text{A})[(x-h)^2+(y-k)^2] = (lx+my+n)^2$,

if (h, k) be a focus, and $lx+my+n=0$ the corresponding directrix. The equations of identity are $a = l^2-\text{A}+2h$, $b = lm+k$, $0 = m^2-\text{A}$, $g = ln+\text{A}h-\frac{1}{2}(h^2+k^2)$, $f = mn+\text{A}k$ and $0 = n^2-\text{A}(h^2+k^2)$, which lead to

$$\text{A}^2(\text{A}^2+a\text{A}-2g) = (f-b\text{A})^2, \quad \text{A}^2h^2+2\text{A}fk-f^2 = 0,$$

$$k^2-2bk+2\text{A}h+b^2-\text{A}(a+\text{A}) = 0,$$

so that four foci lie on each of the circles

$$\text{A}^2(h^2+k^2)+2(f-\text{A}b)\,\text{A}k+2\text{A}^3h-2g\text{A}^2+2bf\text{A}-2f^2 = 0,$$

and concyclic foci correspond to one value of A. Consequently

$$p : q : r : s = lx+my+n : l_1x+my+n_1 : \&c.,$$

and $p-q \propto (l-l_1)x+(n-n_1)$, or as $(k_1-k)\left(\frac{x}{\text{A}}+1\right)$, [consequently $(k_2-k_1)p+(k-k_2)q+(k_1-k)v = 0$, &c., the well known relations of the focal distances,] and

$$(p-q)(q-r)(r-p) \propto (k_1-k)(k_2-k_1)(k-k_2)\left(\frac{x}{\text{A}}+1\right)^3 \propto \text{ABC}\left(\frac{x}{\text{A}}+1\right)^3,$$

since (h, k), (h_1, k_1), (h_2, k_2) lie on the second of the above parabolas, therefore

$$\frac{(p-q)(q-r)(r-p)}{\triangle\text{ABC}} = \frac{(q-r)(r-s)(s-q)}{\triangle\text{BCD}} = \&c.$$

If two of the connectors of the four concyclic foci be parallel, the equation $\lambda(A^2h^2+2Afk-f^2)+[k^2-2bk+2Ah+b^2-A(a+A)]=0$, involves $\lambda=0$ and $A=0$, and reduces to $(k-b)^2=0$, and A, C and B, D become two pairs of coincident points.

With the fuller form $x(x^2+y^2)=ax^2+2bxy+2fy+2gx+c$ for the circular cubic, the quartic in A and the equations to the parabolas and the circle become $A^4+aA^3-2gA^2+cA=(f-bA)^2$;

$$A^2h^2+2Afk-f^2+Ac=0,\quad k^2-2bk+2Ah+b^2-A(a+A)=0;$$

$$A^2(h^2+k^2)+2Ak(f-bA)+2A^2h-2gA^2+2hfA-2f^2+2Ac=0 \ldots\ldots (1).$$

Now any bicircular quartic may be inverted into a circular cubic of this form, and if $\xi=\frac{h}{h^2+k^2}$, $\eta=\frac{k}{h^2+k^2}$, the equivalent equation

$$(x+A)[(x-h)^2+(y-k)^2]=(lx+my+n)^2$$

becomes $$\frac{x+A(x^2+y^2)}{\xi^2+\eta^2}\{(x-\xi)^2+(y-\eta)^2\}=\{lx+my+n(x^2+y^2)\}^2,$$

so that (ξ, η) is a focus, and the foci lie by fours on the circles inverse to (1), which are four in number, one for each value of A. For concyclic focal distances p, q, r, s,

$$\frac{p}{(\xi^2+\eta^2)^{\frac{1}{2}}}:\frac{q}{(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}=lx+my+n(x^2+y^2):l_1x+m_1y+n_1(x^2+y^2)$$

$$\frac{p}{(\xi^2+\eta^2)^{\frac{1}{2}}}-\frac{q}{(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}:\frac{q}{(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}-\frac{r}{(\xi_2^2+\eta_2^2)^{\frac{1}{2}}}:\text{ \&c.}$$

$$=\frac{\eta_1}{(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}-\frac{\eta}{(\xi^2+\eta^2)^{\frac{1}{2}}}:\frac{\eta_2}{(\xi_2^2+\eta_2^2)^{\frac{1}{2}}}-\frac{\eta_1}{(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}:\text{ \&c.};$$

therefore $Ap+Bq+Cr=0$, &c., as is well known.

[Prof. SYLVESTER remarks that the two circular cubics belonging to a concyclic system of points A, B, C, D, are the loci of the two systems of foci of conics passing through the points; and when AC and BD become two pairs of coincident points, the second cubic becomes a circle and a straight right line. See the solutions of Quest. 1900, given on various pages of Vol. VI. of the *Reprints*, especially Art. 5 of Prof. CAYLEY'S solution of the Question, where it is shown that in analytical sense, the degenerate circular cubic is a pair of coincident lines and the line at infinity.]

III. *Solution by* Professor NASH, M.A.

6405. If A, −B, C, −D denote the areas of the four triangles BCD, CDA, DAB, ABC (ABCD being in order on the circle), the given relations are equivalent to the three following equations:—

$$A+B+C+D=0,\quad Ap+Bq+Cr+Ds=0 \ldots\ldots\ldots\ldots (1, 2),$$

$$Ap^2+Bq^2+Cr^2+Ds^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Equation (1) is evident; and (3) is satisfied if p, q, r, s be the distances from A, B, C, D of any point in the plane. For, if $(x_1 y_1)$ $(x_2 y_2)$, &c. be the coordinates of these points, and r the radius of the circle, the centre being origin, we have

$$Ap^2+Bq^2+Cr^2+Ds^2=(x^2+y^2+r^2)(A+B+C+D)$$

$$-2x(x_1A+x_2B+x_3C+x_4D)-2y(y_1A+y_2B+y_3C+y_4D) \ldots\ldots (4);$$

and since $A = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$ with similar expressions for B, C, D, the coefficient of $2x$ in (4) is $\begin{vmatrix} x_1 & x_1 & y_1 & 1 \\ x_2 & x_2 & y_2 & 1 \\ x_3 & x_3 & y_3 & 1 \\ x_4 & x_4 & y_4 & 1 \end{vmatrix} = 0,$

Similarly the coefficient of y vanishes; hence, therefore, (3) is satisfied.

It remains to show that (2) is satisfied for any point on either of the two circular cubics having A, B, C, D as foci.

By means of (1) and (3), the equation (2) may be written

$$(Ap + Bq + Cr)^2 = (A + B + C)(Ap^2 + Bq^2 + Cr^2), \text{ or}$$

$$A(B+C)p^2 + B(C+A)q^2 + C(A+B)r^2 - 2BCqr - 2CArp - 2ABpq = 0 \ldots (5).$$

The discriminant of this equation vanishes, and hence the equation is equivalent to two equations of the form $lp + mq + nr = 0$. The two equations are $A(B+C)p - [AB \pm (ABCD)^{\frac{1}{2}}]q - [AC \mp (ABCD)^{\frac{1}{2}}]r = 0.$

Since the sum of the coefficients = 0, these equations represent two circular cubics having A, B, C as foci. Since we might have eliminated one of the other focal distances instead of s, D of course is also a focus. Hence (2) is satisfied for both of the circular cubics having A, B, C, D as foci. Therefore, &c.

6531. Suppose AD parallel to BC, then $A + D = 0$, $B + C = 0$, and (5) breaks up into the two factors

$$q - r = 0, \quad \text{and} \quad 2Ap - (A-B)q - (A+B)r = 0.$$

The first gives the straight line bisecting AD and BC, the second gives a circular cubic having this line at axis, the collinear foci being the anti-points of the pairs AD and BC.

In order to find the true meaning of the solution $q - r = 0$ we must take a different definition of a circular cubic.

If we draw the two parabolas p, p' through the four concyclic foci A, B, C, D, the circular cubics are the envelopes of the two systems of circles having their centres in p, p' and cutting the circle orthogonally. Now, if AD be parallel to BC, one of the parabolas degenerates into the pair of parallel lines AD and BC, and the envelope becomes the two pairs of anti-points of AD and BC, and the circular points at infinity.

6150. (By H. W. Harris, B.A.) — In the *Quarterly Journal of Mathematics*, Vol. I., Prof. Cayley has discussed the locus of the vertex of a triangle circumscribing a given conic and whose vertices move on given curves. In the case of the curves being both conics, the locus is of the eighth degree. Show that, in the case of all three curves being parabolas inscribed in the same triangle, the locus will reduce to a conic; and show how this last is related to the other three.

Solution by the Proposer.

It is easily proved (see Salmon's *Conics*, 5th ed., p. 359) that the locus of the *free* vertex of a triangle which circumscribes a conic, and two of whose

vertices move on confocal conics, is another conic confocal with the rest. Projecting this theorem, we have a corresponding one for conics inscribed in the same quadrilateral, and projecting one side of this quadrilateral we have the corresponding theorem for parabolas inscribed in the same triangle, and of course the conic which is the locus of the free vertex is another parabola inscribed in the same parabola. Also, since the property above of confocal conics is true for any polygon, the corresponding property of the parabolas is also true generally; that is to say, if a polygon of n sides be such that it circumscribes, while $n-1$ of its vertices describe each a parabola belonging to a system having three tangents common, then the locus of the free vertex is a similar parabola.

It may be noted that a similar proposition is true where the system of parabolas are replaced by a system of conics having double contact with each other along the same line. This is evident by projecting from the case of concentric circles, which may either be regarded as evident or as a particular case of the theorem regarding confocal conics, when the eccentricity vanishes.

6292. (By J. McDowell, M.A.) — If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \ldots + \frac{1}{n}(-1)^{n-1}p_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}.$$

6323. (By the Rev. D. Thomas, M.A.)—If n_m denote the coefficient of x^m in the expansion of $(1+x)^n$, where n is a positive integer, and if

$$F_r(n) = \frac{n_r}{1} - \frac{n_{r+1}}{2} + \ldots;\ \text{prove that } F_r(n) = n_{r-1}\left\{\frac{1}{r} + \ldots + \frac{1}{n}\right\}.$$

6324. (By J. J. Walker, M.A.)—If m, n be positive integers, or which m is not less than n, and p_r, q_r stand for the coefficients of x^r in the development of $(1+x)^n$, $(1-x)^{-m+n-1}$ respectively, prove that

$$\frac{p_1}{q_1} - \frac{p_2}{2q_2} + \frac{p_3}{3q_3} \ldots (-1)^{n-1}\frac{p_n}{nq_n} = \frac{1}{m-n+1} + \frac{1}{m-n+2} + \ldots + \frac{1}{n}.$$

Solution by J. J. Walker, M.A.

Mr. McDowell's equality was given as Quest. 2928 by Mr. Artemas Martin (*Reprint*, Vol. XII., pp. 80—83), and very fully discussed by Mr. Blissard, while Mr. Laverty, Prof. Wolstenholme, and others contributed an inductive solution. Before I had (quite accidentally) met with the earlier proposal of the question, it interested me, as suggesting a second form of a familiar successive differential coefficient; viz., if $u = x^m \log_e x$, then, by Leibnitz's Theorem ($n \not> m$), we have

$$\frac{d^n u}{dx^n} = x^{m-n}\left\{m \ldots (m-n+1)\log_e x + p_1 m \ldots (m-n+2) - p_2 m \ldots (m-n+3) + 2p_3 m \ldots (m-n+4) + 2 \cdot 3 p_4 m \ldots (m-n+5) - \ldots\right\};$$

or $\dfrac{1}{m\ldots(m-n+1)}\dfrac{d^n u}{dx^n} = x^{m-n}\left\{\log_e x + \dfrac{p_1}{q_1} - \tfrac{1}{2}\dfrac{p_2}{q_2} + \tfrac{1}{3}\dfrac{p_3}{q_3}\ldots(-1)^{n-1}\dfrac{1}{n}\dfrac{p_n}{q_n}\right.$.

But, by another method of differentiating (see Quest. 6552),

$$\frac{1}{m\ldots(m-n+1)}\frac{d^n u}{dx^n} = x^{m-n}\left(\log_e x + \frac{1}{m-n+1} + \frac{1}{m-n+2} + \ldots + \frac{1}{m}\right).$$

Thus the equality in Quest. 6324 is established, and, by making $m = n$, that given in Quests. 2928 and 6222 follows.

Writing $m-n+1 = r$ or $n = m-r+1$, it is easily seen that the equality of 6324 is transformed into what that of 6323 would become after division by n_{r-1}, and substitution of m for n.

6389. (By J. W. Russell, M.A.)—A conic cuts the sides of a triangle ABC in the pairs of points a_1a_2, b_1b_2, c_1c_2 respectively: if Bb_2, Cc_2 intersect in a_1; Bb_1, Cc_2 in a_2, and so on; and if $\beta_1\beta_2\beta_3\beta_4$, $\gamma_1\gamma_2\gamma_3\gamma_4$ be similarly constructed; show that the straight lines obtained by putting in various suffixes in $A\alpha$, $B\beta$, $C\gamma$ meet three by three in eight points.

Solution by W. H. Blythe, B.A.; W. B. Grove, B.A.; *and others.*

First, to prove that, if a conic cut the sides of a triangle in a_1a_2, b_1b_2, c_1c_2, then the six straight lines Aa_1, Aa_2, Bb_1, Bb_2, Cc_1, Cc_2 touch the same conic.

We have, by Carnot's theorem,

$Ac_1Ac_2 \cdot Ba_1Ba_2 \cdot Cb_1Cb_2 = Ab_1Ab_2 \cdot Ca_1Ca_2 \cdot Bc_1Bc_2$.

Hence, by Trigonometry,

$\sin ACc_1 \cdot \sin ACc_2 \cdot$ &c. $= \sin ABb_1 \cdot \sin ABb_2$, &c.

Hence, by Salmon's *Conics*, Art. 313, the proposition follows.

Hence $A\beta Ca B\gamma$ is a Brianchon's hexagon; hence $A\alpha$, $B\beta$, $C\gamma$ meet in a point O.

Now α can be chosen in four ways, β then in two ways, and γ then in one way. Hence there are 8 points such as O.

6102. (By the Editor).—If a coin be thrown at random on a horizontal plane; show that (1) the probability of its reaching the plane at an inclination between α and β degrees is $\cos\alpha - \cos\beta$; and (2) the average of all the inclinations with which the coin may reach the plane is the angle subtended by an arc equal to the radius.

Solution by Prof. SEITZ, M.A.; *the* Rev. T. R. TERRY, M.A.; *and others.*

1. The number of ways the coin can reach the plane with the inclination θ is $2\pi \sin\theta$; hence the probability that the inclination will be between α and β degrees is

$$\int_{\alpha}^{\beta} 2\pi \sin\theta\, d\theta + \int_{0}^{\frac{1}{2}\pi} 2\pi \sin\theta\, d\theta = \int_{\alpha}^{\beta} \sin\theta\, d\theta = \cos\alpha - \cos\beta.$$

2. The average of all the inclinations is

$$\int_{0}^{\frac{1}{2}\pi} \theta \,.\, 2\pi \sin\theta\, d\theta + \int_{0}^{\frac{1}{2}\pi} 2\pi \sin\theta\, d\theta = \int_{0}^{\frac{1}{2}\pi} \theta \sin\theta\, d\theta = 1,$$

the length of the arc which measures the inclination, $= 57° 17' 44''{\cdot}8$.

[The above average shows that, if the orbits of comets be supposed to be equally distributed through space, their mean inclination to the plane of the ecliptic is the angle subtended by an arc equal to the radius; a theorem whereon LIAGRE (*Calcul des Probabilités*, Art. 113) has the following note:—"Si l'on calcule, pour les comètes connues jusqu'aujourd'hui, la moyenne de l'inclinaison de leurs orbites sur l'écliptique, on trouve une valeur d'environ 50°. On doit donc conclure de ce fait que les comètes ont une tendance à *se rapprocher* du plan de l'écliptique, conclusion contraire à celle qu'a tirée Laplace" (*Théorie analytique des Probabilités*, p. 259.)

The given Question (6102) is the same in substance as the following Question (567) of the *Nouvelles Annales de Mathématiques*:—"Quelle au la probabilité que l'angle aigu formé par deux grands cercles tracés tes hasard sur une sphère soit comprise entre m degrés et n degrés?" Of this Question there is a solution in the *Annales* for August, 1861; where the probability is found to be $\cos m - \cos n$; but it is remarked that "Laplace trouve ($\cos m - \cos n$: 90°), ce qui semble inexact."]

5878 (By Prof. WOLSTENHOLME, M.A.)—The four real foci of an axial circular cubic are A, B, C, D, and the foot of the asymptote is O: O is the centroid of A, B, C, D, and if α, β, γ, δ be the algebraical distances, OA, OB, OC, OD, the vector equation of the cubic (to A, B, C) is $(\beta^2-\gamma^2)\,AP + (\gamma^2-\alpha^2)\,BP + (\alpha^2-\beta^2)\,CP = 0$. Also, if the vertices of the cubic be U, V, W, a Cartesian whose foci are U, V, W will have its vertices at A, B, C, D, and its centre (or triple focus) at O. In a Cassinoid in which, at any point P, $AP \,.\, BP = m \,.\, OA^2$, if A′, B′ be the other two axial foci, $A'P \,.\, B'P = m \,.\, OP^2$, O being the centre.

Solution by Prof. NASH, M.A.; Prof. EVANS, M.A.; *and others.*

The equation of an axial circular cubic, the asymptote being the axis of y, must be of the form $x(x^2+y^2) + ax^2 + bx + c = 0$ .....(1).

Putting $y = (-1)^{\frac{1}{2}}(x-z)$ and finding the condition for equal roots, the four axial foci are determined by

$$z^4 - 2bz^2 - 8cz + b^2 - 4ac = 0 \text{ .....(2).}$$

Since this equation has no term z^3, $\Sigma z = \alpha + \beta + \gamma + \delta = 0$, and O is the

centroid of A, B, C, D. If the vector equation referred to ABC be $l \cdot AP + m \cdot BP + n \cdot CP = 0$, equation (1) must be identical with

$$l[(x-a)^2+y^2]^{\frac{1}{2}} + m[(x-\beta)^2+y^2]^{\frac{1}{2}} + n[(x-\gamma)^2+y^2]^{\frac{1}{2}} = 0,$$

or $l^4(r^2-2ax+a^2)^2 + m^4(r^2-2\beta x+\beta^2)^2 + n^4(r^2-2\gamma x+\gamma^2)^2$

$$= 2m^2n^2(r^2-2\beta x+\beta^2)^2(r^2-2\gamma x+\gamma^2)+\ldots \quad \ldots\ldots(3),$$

where $r^2 = x^2+y^2$.

The terms r^4 and r^2 do not appear in (1), therefore, from (3 and 4),

$$\Sigma l^4 = 2\Sigma m^2n^2, \quad \Sigma l^4a^2 = \Sigma m^2n^2(\beta^2+\gamma^2), \quad l \pm m \pm n = 0 \ldots\ldots\ldots\ldots(4, 5, 6).$$

If we suppose AP, BP, CP to contain their signs implicitly, we may take the form (6) as $l+m+n=0$; and (5) may be written in the form

$$l^2a^2(m^2+n^2-l^2)+m^2\beta^2(n^2+l^2-m^2)+n^2\gamma^2(l^2+m^2-n^2) = 0.$$

Substituting $-(m+n)$ for l, we get $\frac{m}{\gamma^2-\alpha^2} = \frac{n}{\alpha^2-\beta^2} = \frac{l}{\beta^2-\gamma^2}$, therefore vector equation is $(\beta^2-\gamma^2)AP + (\gamma^2-\alpha^2)BP + (\alpha^2-\beta^2)CP = 0$ .....(7).

Again, if the distances of the foci of a Cartesian from the triple focus be the roots of the equation $z^3-pz^2+qz-r=0$ .....(8), the equation of the Cartesian is $(x^2+y^2-q)^2+4r(2x-p)=0$, the triple focus being the origin; and if the origin be at a distance k from the triple focus, the equation is $[(x+k)^2+y^2-q]^2+4r(2x+2k-p)=0$. The vertices of this Cartesian are given by

$$(x+k)^4-2q(x+k)^2+8r(x+k)+q^2-4pr = 0 \quad \ldots\ldots\ldots(9).$$

If these vertices are the foci of the cubic, (2) and (9) are identical, therefore $k=0$, $b=q$, $c=-r$, $a=-p$. Therefore O the origin is the triple focus, and the single foci are given by the equation (8), $x^3+ax^2+bx+c=0$. But, from (1), this is the equation determining the vertices U, V, W of the cubic.

This result might also be deduced from the vector equations, since from the system $l \cdot AU + m \cdot BU + n \cdot CU = 0$, &c., we can obtain another system of the form $\lambda \cdot AU + \mu \cdot AV + \nu \cdot AW = 0$. For the Cassinian, if $OA = a$,

$$[(x-a)^2+y^2][(x+a)^2+y^2] = m^2a^4, \text{ or } (r^2+a^2)^2-4a^2x^2 = m^2a^4 \ldots\ldots\ldots(\alpha).$$

The other foci A′, B′ are at a distance from $O = a(1-m^2)^{\frac{1}{2}}$. Therefore

$$A'P^2 \cdot B'P^2 = [r^2+a^2(1-m^2)]^2-4a^2x^2(1-m^2)$$

$$= (r^2+a^2-a^2m^2)^2-(1-m^2)[(r^2+a^2)^2-m^2a^4], \text{ by } (\alpha),$$

$$= m^2(r^2+a^2)^2-2m^2a^2(r^2+a^2)+m^2a^4 = m^2r^4.$$

Hence we have $A'P \cdot B'P = m \cdot OP^2$.

4753. (By Prof. Crofton, F.R.S.)—If 1, 2, 3, 4 are four conclyclic points on a Cassinian oval whose foci are F, F′, show that a bicircular oval can be drawn with 1, 2, 3, 4 as foci to pass through F, F′. Show also that the tangents at F, F′ are *double tangents;* each touching the bicircular oval in a second point, and each passing through the centre of the circle 1234.

Solution by Prof. NASH, M.A.; Prof. SCHEFFER, M.A.; *and others.*

The equation of a Cassinian is $(x^2+y^2+c^2)^2-4c^2x^2=m^4$ ..... (1), the foci of which are $x=\pm c$, $y=0$. Let the equation of the circle through 1, 2, 3, 4 be $x^2+y^2=2\alpha x+2\beta y+\gamma$ ..... (2), α, β being the coordinates of the centre O. Substituting for x^2+y^2 in (1), the equation of a conic through 1, 2, 3, 4 is

$$(2\alpha x+2\beta y+\gamma+c^2)^2-4c^2x^2=m^4 \quad \text{..... (3).}$$

The asymptotes of this conic are parallel to the lines $(\alpha x+\beta y)^2-c^2x^2=0$, and therefore perpendicular to the lines OF, OF', *i.e.*,

$$(\beta x-\alpha y)^2-c^2(y-\beta)^2=0.$$

Hence OF, OF' are double tangents to a bicircular quartic, having (2) for a circle of inversion, and (3) for the corresponding focal conic. [See Prof. CASEY'S paper on *Bicircular Quartics*, Art. 47.] It remains to show that F, F' are points of contact. If p be the perpendicular from O upon the asymptote, and r the distance from O of a point of contact of the double tangent OF; then, since the two points of contact are inverse with respect to (2), and are also equidistant from the asymptote $2pr=r^2+\alpha^2+\beta^2+\gamma$. Now, taking the asymptote $2\alpha x+2\beta y+\gamma+c^2-2cx=0$,

$$p=\frac{2\alpha^2+2\beta^2+\gamma+c^2-2c\alpha}{2\left[(\alpha-c)^2+\beta^2\right]^{\frac{1}{2}}}=\frac{OF^2+\alpha^2+\beta^2+\gamma}{2OF};$$

hence F is one point of contact of the double tangent. The latter part may be otherwise proved as follows:—The sixteen points of contact of the eight double tangents of a bicircular quartic lie by fours upon four concentric circles; the common centre being the centre of the focal conics, and each circle cuts orthogonally one of the circles of inversion. Hence the points of contact of the double tangents OF and OF' lie upon the circle

$$x^2+\left(y+\frac{c^2+\gamma}{2\beta}\right)^2=\left(\frac{c^2+\gamma}{2\beta}\right)^2+c^2+\gamma-\gamma, \text{ or } x^2+y^2+\frac{c^2+\gamma}{\beta}y=c^2,$$

and this circle evidently passes through F, F'.

6554. (By C. E. BICKMORE, M.A.)—Solve the equations

$$xyz+x-y-z=a(1+yz-zx-xy),\quad xyz+y-z-x=b(1+zx-xy-yz),$$
$$xyz+z-x-y=c(1+xy-yz-zx).$$

Solution by Rev. J. L. KITCHIN, M.A.; J. O'REGAN; *and others.*

The equations are equivalent to

$$\frac{1-y}{1+y}\cdot\frac{1-z}{1+z}=\frac{1+a}{1-a}\cdot\frac{1-x}{1+x}, \text{ \&c. \&c.;}$$

hence $$\frac{1-x}{1+x}\left(\frac{1+a}{1-a}\right)^{\frac{1}{2}}=\frac{1-y}{1+y}\left(\frac{1+b}{1-b}\right)^{\frac{1}{2}}=\left(\frac{1-z}{1+z}\right)\left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}$$
$$=\left(\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\cdot\frac{1+c}{1-c}\right)^{\frac{1}{2}}.$$

5083 & 5180. (By Prof. WOLSTENHOLME, M.A.)—A bicircular quartic whose real foci lie on a circle is the locus of a point P, which moves so that the ratio AP . BP : CP . DP is constant, A, B, C, D being four fixed points on the focal circle; and there are for each curve two sets of such fixed points. Given the foci of the curve, show how to obtain these points; or, given one set of the points, show how to obtain the other set and the foci.

A circular cubic whose four real foci lie on a circle is the locus of a point P, which moves so that AP . BP = CP . DP, where A, B, C, D are four fixed points on the focal circle, and there are for each curve two sets of such points. Given the foci of the curve, show how to obtain these points; or, given one set of the points, show how to obtain the other set and the foci.

Solution by Prof. NASH, M.A; CHRISTINE LADD; *and others.*

It will be sufficient to prove Quest. 5180, as Quest. 5083 can at once be deduced by inversion from any point.

To find the locus of a point P which moves so that AP . BP = CP . DP, A, B, C, D being four given points on a circle. Here P is evidently the locus of the intersection of the circles AP = k . CP, k . BP = DP. Now, k being variable, AP = k . CP represents a system of circles coaxal with the point circles A and C, and therefore orthotomic to the given circle, and similarly for k . BP = DP. Also, if Q, R be the centres of corresponding circles, we must have AQ : CQ = DR : BR; hence the centres Q, R divide AC and DB proportionately; therefore QR envelopes a parabola touching AC and BD.

If P, P′ be the two points of intersection of these circles, since they both cut the circle ABCD orthogonally, P, P′ must pass through its centre O, and are also inverse points with respect to this circle. Therefore P, P′ lie upon a circular cubic, having as foci the four points in which the parabola cuts the given circle. [See Prof. CASEY'S paper on *Bicircular Quartics* in the Royal Irish Academy's *Transactions*, Vol. 24, p. 492.]

Taking the values $k = 0$, $k = \infty$, we see that AD, BC are tangents to the parabola. If the foci are given, the problem becomes—To inscribe in a given circle a quadrilateral whose sides shall touch a given parabola; a problem which is not generally possible, except when the inverse of the focus with respect to the circle lies upon the directrix. If we take the particular case when the four foci are collinear, the equation of the cubic is $x(x^2+y^2)+ax^2+bx+c=0$; and, in order that this may be the locus of a point P satisfying the relation AP . BP = CP . DP, where A, B, C, D are points on the axis of the curve, we must have the relation

$$a^3-4ab+8c=0.$$

The other cases can be deduced from this by inversion.

5773. (By J. L. MCKENZIE, B.A.) — Through any point P on a circular cubic draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P.

Solution by Prof. NASH, M.A.; Prof. MATZ, M.A.; *and others.*

Using the notation (AB) to denote the third point in which the line AB meets the cubic, we have [(AB), (CP)] and also [(AB), (DE)] each parallel to the asymptote; hence CP, DE meet the curve in the same point K, or more briefly (CP) = (DE) = K.

Again, D, E, Q, R being four points on the cubic, we have

$$[(DE)(QR)] = [(DQ)(RE)],$$

since the pairs of lines DE, QR and DQ, RE may be considered as two conics through the four points; therefore (KC) = (PS); but (KC) = P, therefore (PS) = P, *i.e.* the line PS meets the curve again at P, and is therefore the tangent at P.

6563. (By Prof. SYLVESTER, F.R.S.) — 1. The velocities of three bodies in the direction of the lines joining them with their centre of gravity, and the lengths of these lines, being given; find the transverse velocities [that is, perpendicular respectively to the former ones] in the plane of the bodies, in order that the centre of gravity, and the principal axes through it of the system, may remain at rest for a moment of time.

2. Given the distances of three bodies from their centre of gravity, determine the angles which they make respectively with the principal axes of the system, and the values of the moments of inertia in respect to these axes.

Solution by J. J. WALKER, M.A.

By "bodies" is to be understood masses (m, m', m'') whose linear dimensions are infinitesimal in comparison with their distances (say a, b, c) from their common centre of gravity (G). Let θ, θ', θ'' be the angles which these distances respectively make with a principal axis through G, in the plane of the three bodies; then the following relations hold:—

$$ma^2 \sin 2\theta + m'b^2 \sin 2\theta' + m''c^2 \sin 2\theta'' = 0 \quad \ldots\ldots(1),$$

$$ma \sin \theta + m'b \sin \theta' + m''c \sin \theta'' = 0 \quad \ldots\ldots(2),$$

$$ma \cos \theta + m'b \cos \theta + m''c \cos \theta'' = 0 \quad \ldots\ldots(3).$$

Squaring (2), (3), after transferring the last terms to the other side, and adding, we get

$$\cos(\theta - \theta') = \cos\gamma = (-m^2a^2 - m'^2b^2 + m''^2c^2) : 2mm'bc,$$

giving $\sin(\theta - \theta') = \sin\gamma = (2\Sigma m'^2m''^2b^2c^2 - \Sigma m^4a^4)^{\frac{1}{2}} : 2mm'bc$;

with similar values for $\cos(\theta - \theta'') = \cos\beta$, $\sin(\theta - \theta'') = \sin\beta$. Writing $\theta + \gamma$ for θ', $\theta + \beta$ for θ'' in (1), and dividing by $\cos 2\theta$, θ or $\frac{1}{2}\pi + \theta$ is determined by

$$\tan 2\theta = -(+m'b^2 \sin 2\gamma + m''c^2 \sin 2\beta) : (ma^2 + m'b^2 \cos 2\gamma + m''c^2 \cos 2\beta),$$

and thus the second part of the question is solved. For the first part, let δa, $a\delta\theta \ldots$ be the displacements along, and transversely to, $a \ldots$ in the element of time δt; then G and the principal axes will plainly be unmoved if equations (1), (2), (3) hold after substitution of $a + \delta a$, $\theta + \delta\theta \ldots$ for a, $\theta \ldots$. This condition leads to

$$ma^2 \cos 2\theta \,.\, \delta\theta + m'b^2 \cos 2\theta' \,.\, \delta\theta' + m''c^2 \cos 2\theta'' \,.\, \delta\theta'' = -a\delta a \sin 2\theta - \ldots \quad \ldots(\text{i.})$$

$$ma \cos\theta \,.\, \delta\theta + m'b \cos\theta' \,.\, \delta\theta' + m''c \cos\theta'' \,.\, \delta\theta'' = -\delta a \sin\theta - \ldots \quad \ldots(\text{ii.})$$

$$ma \sin\theta \,.\, \delta\theta + m'b \sin\theta' \,.\, \delta\theta' + m''c \sin\theta'' \,.\, \delta\theta'' = \delta a \cos\theta + \ldots \quad \ldots(\text{iii.}),$$

equations which determine $a\delta\theta$, $b\delta\theta'$, $c\delta\theta'' : \delta t$, in terms of

$$v = \delta a : \delta t, \quad v' = \delta a' : \delta t, \quad v'' = \delta a'' : \delta t.$$

[Prof. SYLVESTER remarks that the above questions have an important application in the method employed by him to throw into equations the problem of the motion of three bodies acted on by their mutual attractions, which he reduces to a system of three differential equations of the second order, and one of the first, between the time and four quantities, viz., the distances from the centre of gravity and the angular velocity of the principal axis, equivalent (leaving out t) to six linear simultaneous equations between seven quantities (p, q, r ; p', q', r' ; ω).

JACOBI, in the "Elimination of the Nodes," reduces the problem to depend on five equations of the first order and one of the second, of course equivalent (leaving out t) to six linear simultaneous equations between seven variables. JACOBI transforms the problem, at starting, into one relating only to two bodies; but Prof. SYLVESTER adheres to the original data throughout, thus adopting a method which he believes to be more natural, and leading to a result superior in point of form.]

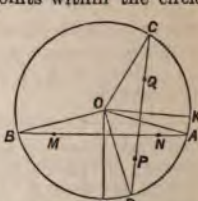
5533. (By the EDITOR.)—Through two points taken at random anywhere inside a circle a chord is drawn, and a second chord through two other such random points: show that (1) the probability that these two chords will intersect inside the circle is $\frac{1}{3} + \frac{245}{72\pi^2}$; also (2), if through a third pair of such random points, a third chord be drawn, show that the respective probabilities that the three chords will intersect inside the circle in 0, 1, 2, 3 points are

$$p_0 = \frac{1}{3} - \frac{245}{36\pi^2} + \frac{23023}{576\pi^4}, \quad p_1 = \frac{2}{5} + \frac{155}{36\pi^2} - \frac{55055}{864\pi^4},$$

$$p_2 = \frac{1}{5} + \frac{115}{72\pi^2} + \frac{13013}{1728\pi^4}, \quad p_3 = \frac{1}{15} + \frac{65}{72\pi^2} + \frac{7007}{432\pi^4}.$$

Solution by Professor SEITZ, M.A.

1. Let M, N and P, Q be two pairs of random points within the circle whose centre is O, and AB, CD the chords drawn through them. Draw the radii OH, OK perpendicular to AB, CD. Let OA $= r$, AM $= u$, MN $= v$, CP $= w$, PQ $= x$, AB $= u'$, CD $= w'$, $\angle$ AOH $= \theta$, $\angle$ COK $= \phi$, $\angle$ HOK $= \mu$, and ω = the angle OH makes with a fixed radius. Then $u' = 2r \sin\theta$, $w' = 2r \sin\phi$; an element of the circle at M is $r \sin\theta \, d\theta \, du$, at N it is $d\omega \, v \, dv$, at P it is $r \sin\phi \, d\phi \, dw$, and at Q it is $d\mu \, x \, dx$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ, and θ and $\frac{1}{2}\pi$; of μ, $\theta - \phi$ and $\theta + \phi$ when $\phi < \theta$, and $\phi - \theta$ and



$\phi + \theta$ when $\phi > \theta$, and the result doubled; of ω, 0 and 2π; of u, 0 and u'; of v, 0 and u, and doubled; of w, 0 and w'; and of x, 0 and w, and doubled, Hence, since the whole number of ways the four points can be taken is $\pi^4 r^8$, the required probability is

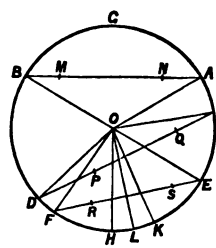
$$p = \frac{8}{\pi^4 r^8} \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} \int_0^{w'} \int_0^{u} \left\{ \int_0^{w'} \int_0^{w} \left[\int_0^{\theta} \int_{\theta-\phi}^{\theta+\phi} r \sin\phi \, d\phi \, d\mu \right.\right.$$

$$\left.\left. + \int_{\theta}^{\frac{1}{2}\pi} \int_{\phi-\theta}^{\phi+\theta} r \sin\phi \, d\phi \, d\mu \right] dw \, x \, dx \right\} r \sin\theta \, d\theta \, d\omega \, du \, v \, dv$$

$$= \frac{64}{3\pi^3} \int_0^{\frac{1}{2}\pi} \left\{ \tfrac{4}{3} \int_0^{\theta} \int_{\theta-\phi}^{\theta+\phi} \sin^4\phi \, d\phi \, d\mu + \tfrac{4}{3} \int_{\theta}^{\frac{1}{2}\pi} \int_{\phi-\theta}^{\phi+\theta} \sin^4\phi \, d\phi \, d\mu \right\} \sin^4\theta \, d\theta$$

$$= \frac{512}{9\pi^3} \int_0^{\frac{1}{2}\pi} \left\{ \int_0^{\theta} \phi \sin^4\phi \, d\phi + \int_0^{\frac{1}{2}\pi} \theta \sin^4\phi \, d\phi \right\} \sin^4\theta \, d\theta$$

$$= \frac{32}{9\pi^3} \int_0^{\frac{1}{2}\pi} (3\pi\theta - 3\theta^2 + 3\sin^2\theta + \sin^4\theta) \sin^4\theta \, d\theta = \frac{1}{3} + \frac{245}{72\pi^2}.$$



2. Let R, S be the third pair of random points, and EF the chord drawn through them. Draw the radius OL perpendicular to EF. Let p_0, p_1, p_2, p_3 be the respective probabilities that AB, CD, EF will intersect in 0, 1, 2, 3 points. Let $ER = y$, $RS = z$, $EF = y'$, $\angle EOL = \downarrow$, and $\angle KOL = \rho$. Then $y' = 2r \sin\downarrow$; an element of the circle at R is $r \sin\psi \, d\downarrow \, dy$, and at S it is $d\rho \, z \, dz$.

For non-intersection, the chord CD being between AB and EF, the limits of θ are 0 and π; of ϕ, 0 and θ; of ψ, 0 and ϕ; of μ, $\phi - \theta$ and $\theta - \phi$; and of ρ, $\psi - \phi$ and $\phi - \psi$. The result of these integrations must be multiplied by 6 to allow for the interchange of the chords.

For non-intersection, the chord EF having its ends in the arc BD, the limits of θ are 0 and π; of ϕ, 0 and θ; of ψ, 0 and $\theta - \phi$; of μ, $2\psi - \theta + \phi = \alpha$ and $\theta - \phi$; and of ρ, $\phi + \psi$ and $\theta + \mu - \psi = \beta$. This result must be multiplied by 4 to allow for the positions of EF, in which its ends are in the arc AC, and for the positions of CD and EF, in which their ends are in the arc AGB. The limits of ω are 0 and π; of u, 0 and u'; of v, 0 and u, and doubled; of w, 0 and w'; of x, 0 and w, and doubled; of y, 0 and y'; and of z, 0 and y, and doubled. Hence, since the whole number of ways the six points can be taken is $\pi^6 r^{12}$, we have

$$p_0 = \frac{48}{\pi^6 r^{12}} \int_0^{\pi} \int_0^{\theta} \int_0^{\phi} \int_{\phi-\theta}^{\theta-\phi} \int_{\psi-\phi}^{\phi-\psi} \int_0^{\pi} \int_0^{u'} \int_0^{u} \int_0^{w'} \int_0^{w} \int_0^{y'} \int_0^{y} r^3 \sin\theta \sin\phi \sin\psi$$

$$\times \, d\theta \, d\phi \, d\psi \, d\mu \, d\rho \, d\omega \, du \, v \, dv \, dw \, x \, dx \, dy \, z \, dz$$

$$+ \frac{32}{\pi^6 r^{12}} \int_0^{\pi} \int_0^{\theta} \int_0^{\theta-\phi} \int_{\theta-\phi}^{\beta} \int_{\phi+\psi}^{\pi} \int_0^{u'} \int_0^{u} \int_0^{w'} \int_0^{w} \int_0^{y'} \int_0^{y} r^3 \sin\theta \sin\phi \sin\psi$$

$$\times \, d\theta \, d\phi \, d\psi \, d\mu \, d\rho \, d\omega \, du \, v \, dv \, dw \, x \, dx \, dy \, z \, dz$$

$$= \frac{1024}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{\int_0^{\phi}\int_{\phi-\theta}^{\theta-\phi}\int_{\psi-\phi}^{\phi-\psi} 3\sin^4\psi\, d\psi\, d\mu\, d\rho + \int_0^{\theta-\phi}\int_{\rho}^{\theta-\phi}\int_{\phi+\psi}^{\beta} 2\sin^4\psi\, d\psi\, d\mu\, d\rho\right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{4096}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{\int_0^{\phi} 3(\theta-\phi)(\phi-\psi)\sin^4\psi\, d\psi + \int_0^{\theta-\phi}(\theta-\phi-\psi)^2\sin^4\psi\, d\psi\right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{16}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{32(\theta-\phi)^3+144\phi^2(\theta-\phi)-60(\theta-\phi)-144(\theta-\phi)\sin^2\phi - 48(\theta-\phi)\sin^4\phi+32\sin 2(\theta-\phi)-\sin 4(\theta-\phi)\right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{1}{81\pi^5}\int_0^{\pi}(360\theta^4-1665\theta^2-1296\theta^2\sin^2\theta-4326^2\sin^4\theta-4020\theta\sin\theta\cos\theta - 408\theta\sin^3\theta\cos\theta+5685\sin^2\theta+559\sin^4\theta + 248\sin^6\theta+36\sin^8\theta)\sin^4\theta\, d\theta$$

$$= \frac{1}{3}-\frac{245}{36\pi^2}+\frac{23023}{576\pi^4}.$$

We will next find the probability that the chords CD, EF will both intersect AB. By reference to the first part of the solution, we readily see that the probability that a third chord will also intersect AB is

$$P = \frac{256}{3\pi^5}\int_0^{\frac{1}{2}\pi}\left\{\tfrac{4}{2}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\sin^4\phi\, d\phi\, d\mu+\tfrac{4}{3}\int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}\sin^4\phi\, d\phi\, d\mu\right\}^2\sin^4\theta\, d\theta$$

$$= \frac{16384}{27\pi^5}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\phi\sin^4\phi\, d\phi+\int_{\theta}^{\frac{1}{2}\pi}\theta\sin^4\phi\, d\phi\right\}^2\sin^4\theta\, d\theta$$

$$= \frac{64}{27\pi^5}\int_0^{\frac{1}{2}\pi}(3\pi\theta-3\theta^2+3\sin^2\theta+\sin^4\theta)^2\sin^4\theta\, d\theta = \frac{2}{15}+\frac{155}{108\pi^2}+\frac{97097}{5184\pi^4}.$$

Now P is evidently equal to the probability that CD, EF will both intersect AB, and not each other, together with p_3; hence we have

$$\tfrac{1}{3}p_2+p_3 = P \quad \text{.....(1).}$$

The probability, p, that AB and CD will intersect, is equal to the probability that they will intersect each other, and not EF, plus the probability that CD, EF will both intersect AB, and not each other, plus the probability that AB, EF will both intersect CD, and not each other, plus p_3; therefore $\frac{1}{3}p_1+\frac{2}{3}p_2+p_3 = p$, also $p_1+p_2+p_3 = 1-p_0$ .....(2, 3). From (1), (2), (3), knowing the values of P, p, p_0, we find for p_1, p_2, p_3 the values given in the question.

5454. (By Prof. Townsend, F.R.S.)—A solid ellipsoid of uniform density being supposed to revolve round its least axis of figure, and to

carry with it a surrounding envelope of homogeneous incompressible fluid of different density, the entire mass attracting according to the ordinary law of the inverse square of the distance; required the several conditions requisite to the permanent assumption of the ellipsoidal form by the free surface of the fluid.

Solution by the PROPOSER.

The entire force acting on any superficial element of the fluid consisting of three parts, arising respectively from—(1) the attraction of the fluid, and of a hypothetical mass, of equal density, occupying the space of the solid; (2) the attraction or repulsion, as the case may be, of a second hypothetical mass, having the difference of densities of the solid and fluid, and occupying again the space of the former; and (3) the rotation of the entire mass, as a whole, round its axis of figure; and the components, parallel to the axes, of the first as well as of the third parts, being, for the assumed form of free surface, proportional to the corresponding coordinates of the element at every point; in order that those of the second part also should be proportional to them, as they must necessarily be for the permanence of the assumed form, the surface of the fluid, on the known principles of the theory of ellipsoidal attraction for the law of the inverse square of the distance, must be confocal with that of the solid; which, accordingly, is the first condition requisite to the possibility of the assumed form.

This condition being supposed fulfilled, and the three semi-axes a_1, b_1, c_1 of the outer being thus completely determined from the three a, b, c of the inner ellipsoid by virtue of the known volume of the fluid; it is further necessary that the external ellipsoid, thus completely determined should be of the Jacobian form,—that, viz., (see BESANT's *Hydromechanics*, 3rd ed., Art. 114,) for which $\int_0^1 \frac{u^2(1-u^2)(1-\lambda^2\mu^2u^2)\,du}{[(1+\lambda^2u^2)(1+\mu^2u^2)]^{\frac{3}{2}}} = 0$;

whence $a_1^2 = c_1^2(1+\lambda^2)$ and $b_1^2 = c_1^2(1+\mu^2)$; which, accordingly, is the second condition requisite to the possibility of the assumed form.

This condition also being supposed fulfilled, it is further requisite, finally, that the velocity of rotation ω have the definite value given (see reference already made) by the equation

$$\omega^2 = \frac{3M}{c_1^2}\int_0^1 \frac{u^2(1-u^2)\,du}{[(1+\lambda^2u^2)(1+\mu^2u^2)]^{\frac{3}{2}}},$$

when M = the entire mass of the solid and fluid taken together; which, accordingly, is the third condition requisite to the possibility of the assumed form.

6565. (By Prof. MINCHIN, M.A.)—Prove that the areas of roulettes, in their most general forms, follow the law of circular transformation obtained by STEINER, for the areas of pedals (see WILLIAMSON's *Integral Calculus*, 3rd ed., p. 202); and thus, in virtue of a general kinematical theorem, deduce at once KEMPE's theorem (*ibid.* p. 210).

Solution by C. LEUDESDORF, M.A.

In the *Messenger* for May, 1878, I gave a proof of this, which I repeat here, as it was not worked out in full, being precisely similar to the proof there given of a theorem relating to areas of pedals.

The most general motion of a point in a plane being reducible to a case of roulettes, let a curve roll upon a fixed curve, carrying points A, B, C, D, which describe, in consequence, areas (A), (B), (C), (D). Let D, referred to ABC, have areal coordinates x, y, z, and let P be the instantaneous point of contact of the two curves. As in WILLIAMSON'S *Integral Calculus*, p. 205,

$$(A) = (S) + \tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)PA^2\,d\omega,$$

with similar equations for B, C, D. Therefore

$$(D)-x(A)-y(B)-z(C) = \tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)(PD^2-xPA^2-yPB^2-zPC^2)\,d\omega$$

$$= -\tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)(a^2yz+b^2zx+c^2xy)\,d\omega$$

(by an easily proved geometrical relation)

$$= k\,(a^2yz+b^2zx+c^2xy),\ k \text{ being a constant.}$$

If (D) is made constant, the locus of D is therefore, in general, a circle; which gives KEMPE'S theorem, as originally deduced by him in the *Messenger* for April, 1878. If, in addition, (A) = (B) = (C), the locus of D is the circle round ABC, as in WILLIAMSON'S *Integral Calculus*, p. 210.

[Prof. MINCHIN states that when, two months ago, he proved and developed this theorem, he looked at the papers referred to without finding any treatment of the results of HOLDITCH, AMSLER, and others, from the Roulette point of view; and he adds that, while the theorems in the *Messenger* are limited to *closed* paths, the above theorem is not so limited.]

6533. (By Prof. CROFTON, F.R.S.)—Prove that

$$(r+1)^r = r^r + 2^0\cdot r(r-1)^{r-1} + 3^1\cdot\frac{r(r-1)}{1\cdot2}(r-2)^{r-2} + 4^2\cdot\frac{r(r-1)(r-2)}{1\cdot2\cdot3}$$

$$\times(r-3)^{r-3} + 5^3\cdot\frac{r(r-1)(r-2)(r-3)}{1\cdot2\cdot3\cdot4}(r-4)^{r-4} + \ldots + (r+1)^{r-1}.$$

Solution by J. J. WALKER, M.A.

In the Mathematical Society's *Proceedings*, Vol. xi., p. 108, I have given the development

$$D^n u\phi^{n+1} = \phi D^n u\phi^n + n\phi'\phi D^{n-1}u\phi^{n-1} + n_2 D\phi'\phi^2\cdot D^{n-2}u\phi^{n-2}\ldots$$

$$\ldots + nD^{n-2}\phi'\phi^{n-1}\cdot Du\phi + uD^{n-1}\phi'\phi^n,$$

u and ϕ being any functions of x, and ϕ' being $D\phi$.

If $u = e^{(r-n)x}$, $\phi = e^x$, both sides being divisible by $e^{(r+1)x}$, we obtain the more general development

$$(r+1)^n = r^n + 2^0 n(r-1)^{n-1} + 3^1 n_2(r-2)^{n-2} + 4^2 n_3(r-3)^{n-3} + \ldots + (n+1)^{n-1};$$

whereof Prof. CROFTON'S is the particular case in which $n = r$ or $u = 1$.

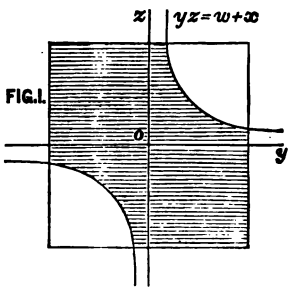
45. (By ELIZABETH BLACKWOOD.)—In the integral

$$\int dw \int dx \int dy \int dz\, \phi\,(w, x, y, z)$$

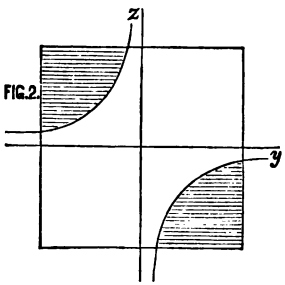
the four variables are each restricted between the limits a and $-a$, and the integration is further restricted by the condition $w + x > yz$; find the limits of integration when z varies first, y next, and w last.

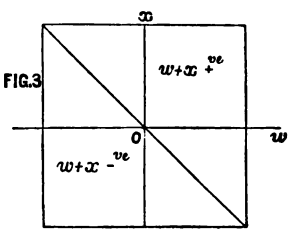
I. *Solution by* G. F. WALKER, M A.; Prof. MATZ, M.A.; *and others.*

Take $w + x$ constant and trace the curve $yz = w + x$; we must take Fig. 1, if $w + x$ be positive, and Fig. 2 if it be negative, the square being $x = \pm a$, $y = \pm b$ in each case, and the point being restricted to the shaded area, and Fig. 3 giving the limits of integration of $w + x$. Let P denote $\phi\,(x, y, z, w)$. The integral with respect to $y + z$ is



$$\int_{-a}^{a}\int_{-a}^{a} dy\,dz\,\mathrm{P} - \int_{\frac{w+x}{a}}^{a}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P} - \int_{-a}^{-\frac{w+x}{a}}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P}$$





if $w + x$ be positive, and $\displaystyle\int_{-\frac{w+x}{a}}^{a}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P} + \int_{-a}^{\frac{w+x}{a}}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P}$

if $w + x$ be negative, and the complete result is

$$\int_{-a}^{a}\int_{-w}^{a}\left[\int_{-a}^{a}\int_{-a}^{a} dy\,dz\,\mathrm{P} - \int_{\frac{w+x}{a}}^{a}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P} - \int_{-a}^{-\frac{w+x}{a}}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P}\right] dw\,dz\,x$$

$$+ \int_{-a}^{a}\int_{-a}^{-w}\left[\int_{-\frac{w+x}{a}}^{a}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P} + \int_{-a}^{\frac{w+x}{a}}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P}\right] dw\,dx.$$

If $a > 2$, this solution is correct. If $a < 2$, we must add

$$\int_{a^2-a}^{a}\int_{a^2-w}^{a}\int_{-a}^{a}\int_{-a}^{a} \mathrm{P}\,dw\,dx\,dy\,dz.$$

II. *Solution by* W. B. Grove, B.A.; H. McColl, B.A.; *and others.*

Let $A \equiv w_{1'.2}\, x_{1'.2}\, y_{1'.2}\, z_{1'.2}\, p\,(w+x-yz)$, where the symbols have the meanings attached to them in Mr. McColl's first Paper in *Proceedings of the London Mathematical Society*. Then, according as $A > < 2$, it will be found that

$$A = w_{1'.2}\,\{x_{1'.3}\,(y_{3'.4}\,z_{1'.2}+y_{4'.2}\,z_{1'.3}+y_{1'.3}\,z_{3'.2}) + x_{3'.2}\,(y_{3'.2}\,z_{1'.3}+y_{1'.4}\,z_{3'.2})\},$$

or
$$A = (w_{1'.3}\,x_{4'.3}+w_{3'.2}\,x_{1'.3})\,(y_{3'.4}\,z_{1'.2}+y_{4'.2}\,z_{1'.3}+y_{1'.3}\,z_{3'.2}) + (w_{1'.4}\,x_{3'.2}+w_{4'.2}\,x_{3'.5})\,(y_{3'.2}\,z_{1'.3}+y_{1'.4}\,z_{3'.2}) + w_{1'.5}\,x_{1'.4}\,y_{1'.2}\,z_{1'.2}.$$

The limits of integration, which in these statements are denoted by suffixes, may be ascertained by referring to the subjoined table:—

$a_1 = 2$	$w_1 = a$ $w_2 = -a$ $w_3 = a^2-a$ $w_4 = a-a^2$	$x_1 = a$ $x_2 = -a$ $x_3 = -w$ $x_4 = a^2-w$ $x_5 = -a^2-w$	$y_1 = a$ $y_2 = -a$ $y_3 = \dfrac{w+x}{a}$ $y_4 = -\dfrac{w+x}{a}$	$z_1 = a$ $z_2 = -a$ $z_3 = \dfrac{w+x}{y}$

6548. (By W. J. C. Sharp, M.A.)—If the normals at four points on an ellipse meet in a point, prove that the sum of the excentric angles at these points is an odd multiple of π.

Solution by Prof. Scott, M.A.; Prof. Matz, M.A.; *and others.*

Three normals meet in a point if $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = 0$, and the fourth goes through intersection of first and second if

$$\sin(\alpha+\delta)+\sin(\beta+\delta)+\sin(\alpha+\beta) = 0;$$

hence, subtracting, $4\cos\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\cos\tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = 0$, or $\alpha+\beta+\gamma+\delta = (2m+1)\pi$.

5309. (By Professor Wolstenholme, M.A.)—The two concentric squares ABCD, A'B'C'D' have their sides parallel, and the side of the larger equal to the sum of side and diagonal of the smaller; prove that (1) an infinite number of sets of four points a, b, c, d can be taken on the

sides of either, such that the lines joining them in order, *ab*, *bc*, *cd*, *da*, pass through the corners of the other [thus, *a*, *b*, *c*, *d* lie on AB, BC, CD, DA, and *ab*, *bc*, *cd*, *da* pass through B′, C′, D′, A′; also a_1, b_1, c_1, d_1 lie on A′B′, B′C′, C′D′, D′A′, and a_1b_1, b_1c_1, c_1d_1, d_1a_1 pass through D, A, B, C]; and (2) *abcd* and $a_1b_1c_1d_1$ are each a mean proportional between the two.

Solution by the PROPOSER.

Area of *abcd* = square A′B′C′D′ + 4 equal triangles

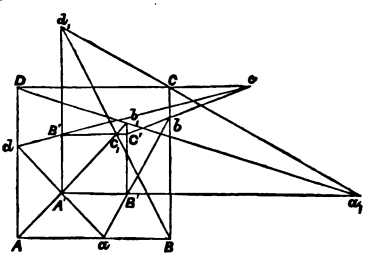
$\left(\text{each} = \text{area of square} \cdot \tfrac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)$

= square A′B′C′D′ $(1-\sqrt{2})$

= mean proportional between the two squares.

So, area $a_1b_1c_1d_1$ = 4 equal $\triangle^s$

$\left(\text{each} = \text{square ABCD} \cdot \tfrac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)$



−square ABCD = square ABCD $(\sqrt{2}-1)$ = square A′B′C′D′ $(\sqrt{2}+1)$ = mean proportional between the two.

The value $\frac{17}{12}$ has been taken in the figure for $\sqrt{2}$, and is sufficiently near. The corresponding system of poristic equations is

$$\frac{1}{x_2}+\frac{1}{1-x_1}=\frac{1}{x_3}+\frac{1}{1-x_2}=\frac{1}{x_4}+\frac{1}{1-x_3}=\frac{1}{x_1}+\frac{1}{1-x_4}=\frac{1}{k},$$

where $k = \frac{1}{2}$, or $(2\pm\sqrt{2})^{-1}$; but the first value reduces the second square to a point, and the in- and circum-scribed figure to a single straight line.

6540. (By Prof. MATZ, M.A.)—A, B, C, D play a set of games in which two are partners against the other two, any two being equally likely to be partners, and drawn games being impossible. If A's chance of winning a single game with B, C, D for partner be p, q, r respectively; find the chances that A, B, C, D respectively will win a out of $a+b$ games.

Solution by W. B. GROVE, B.A.; ELIZABETH BLACKWOOD; *and others.*

Let p_A denote the chance that A wins a single game, $q_A = 1-p_A$, and so on; then $p_A = \frac{1}{3}(p+q+r)$, $p_B = \frac{1}{3}[p+(1-q)+(1-r)]$,

$$p_C = \tfrac{1}{3}[(1-p)+q+(1-r)],\quad p_D = \tfrac{1}{3}[(1-p)+(1-q)+r];$$

and, by a well-known formula, the chance that A wins a games out of $a+b$, with similar expressions for B, C, D, is

$$p_A^a\left\{1+aq_A+\frac{a(a+1)}{1.2}q_A^2+\ldots+\frac{a(a+1)\ldots(a+b-1)}{b!}q_A^b\right\}.$$

5642. (By E. B. ELLIOTT, M.A.)—If $\frac{x-A}{l} = \frac{y-B}{m} = \frac{z-C}{n}$ be the type of a given doubly-infinite set of straight lines, l, m, n being direction-cosines, and A, B, C given functions of these which make $A\,dl + B\,dm + C\,dn$ the exact differential of a function $\phi(l, m, n)$; prove that the polar tangential equation of a class of surfaces cutting all the straight lines orthogonally is $p = \phi(l, m, n) + c$, c being any constant.

Solution by W. J. C. SHARP, B.A.; Prof. JOHNSON, M.A.; *and others.*

Now $\frac{x-A}{l} = \frac{y-B}{m} = \frac{z-C}{n}$ is the equation to one of the lines, and therefore $lx + my + nz = p$ is the equation to the tangent plane to the orthogonal surface, and $(l+\delta l)\,x + (m+\delta m)\,y + (n+\delta n)\,z = p + \delta p$ that to an adjacent tangent plane; hence, at the intersection

$$x\delta l + y\delta m + z\delta n = \delta p,$$

therefore $$A\delta l + B\delta m + C\delta n + k\,(l\delta l + m\delta m + n\delta n) = \delta p,$$

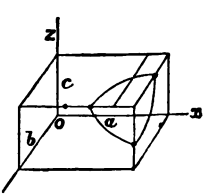
or $$d\,.\,\phi(lmn) + \tfrac{1}{2}kd\,(l^2 + m^2 + n^2) = \delta p;$$

but $l^2 + m^2 + n^2 = 1$, therefore $d\,(l^2 + m^2 + n^2) = 0$, therefore $\phi(lmn) + c = p$.

6523. (By G. F. WALKER, M.A.)—Three quantities lie between $(0, a)$, $(0, b)$, $(0, c)$ respectively; find the chance that their product is less than d^3 $(d^3 < abc)$. [Suggested by Miss BLACKWOOD's Question 6414.]

I. *Solution by* Prof. PRATT, M.A.; *the* PROPOSER; *and others.*

The required chance is the volume of the rectangle abc on the origin side of the surface $xyz = d^3$,



$$\text{and chance} = \frac{abc - \int_{\frac{d^3}{bc}}^{a} \int_{\frac{d^3}{cx}}^{b} y\left(c - \frac{d^3}{xy}\right) dx\,dy}{abc},$$

$$= \frac{d^3}{abc}\left\{1 + \log\frac{abc}{d^3} + \tfrac{1}{2}\left[\log\frac{abc}{d^3}\right]^2\right\}.$$

II. *Solution by* W. B. GROVE, B.A.; H. MCCOLL, B.A.; *and others.*

The chance required $= AQ \iiint dx\,dy\,dz \div A \iiint dx\,dy\,dz$; where x, y, z are the three quantities; A denotes the statement that they lie between $(0, a)$ $(0, b)$ $(0, c)$ respectively; and Q the statement that their product is

less than d^3; therefore

$$A \equiv x_{1'.0}\, y_{1'.0}\, z_{1'.0},$$

$$AQ \equiv x_{1'.0}\, y_{1'.0}\, z_{1'.2'.0}$$

$$= x_{2'.0}\, y_{1'.0}\, z_{1'.0} + x_{1'.2}\, y_{2'.0}\, z_{1'.0} + x_{1'.0}\, y_{1'.2}\, z_{2'.0};$$

$x_1 = a$	$y_1 = a$	$z_1 = c$
$x_2 = \frac{d^3}{bc}$	$y_2 = \frac{d^3}{cx}$	$z_2 = \frac{d^3}{xy}$

hence the chance required is

$$\frac{\int_0^{\frac{d^3}{bc}}\int_0^b\int_0^c dx\,dy\,dz + \int_{\frac{d^3}{bc}}^a\int_0^{\frac{d^3}{xy}}\int_0^c dx\,dy\,dz + \int_0^a\int_{\frac{d^3}{cx}}^b\int_0^{\frac{d^3}{cx}} dx\,dy\,dz}{\int_0^a\int_0^b\int_0^c dx\,dy\,dz}$$

$$= \frac{d^3}{abc}\left\{1 + \log\frac{abc}{d^3} + \tfrac{1}{2}\left(\log\frac{abc}{d^3}\right)^2\right\}.$$

This expression can be extended in a similar way to the case of more than three quantities. If there are n quantities lying between $(0, a)$ $(0, b)$ $(0, c)$... respectively, and d^n be $< abc...$, the chance that their product is less than d^n is

$$\frac{d^n}{abc...}\left\{1 + \log\frac{abc...}{d^n} + \frac{1}{2!}\left(\log\frac{abc...}{d^n}\right)^2 + ... + \frac{1}{(n-1)!}\left(\log\frac{abc...}{d^n}\right)^{n-1}\right\}.$$

6051. (By D. Edwardes.)—A particle describes an ellipse under an attraction to the centre. Prove that (1) the sum of the squares of the velocities at the extremities of a pair of conjugate diameters is constant; (2) the sum of the squares of the reciprocals of the velocities at any two points the directions at which include a right angle is constant.

Solution by the Rev. T. R. Terry, M.A.; D. Wickersham; *and others.*

1. If CP, CD be conjugate diameters, it is known that velocity at P is parallel and proportional to CD. But $CP^2 + CD^2 = \text{constant}$.

2. If CD, Cd are at right angles, $\frac{1}{CD^2} + \frac{1}{Cd^2} = \text{constant}$.

6541. (By Professor Nash, M.A.)—A cardioid being generated by a circle rolling upon a fixed equal circle; prove that the polar reciprocal of the cardioid with respect to the fixed circle is a nodal circular cubic; that there are two real axial foci besides the double focus at the node; and that the vector equation is $3\rho_1 \pm 2\rho_2 - \rho_3 = 0$, where ρ_2 is the distance from the double focus.

I. *Solution by* Prof. WOLSTENHOLME, M.A.; CHRISTINE LADD; *and others.*

The equation of the tangent of the cardioid (origin at centre of fixed circle) is $x \sin 3\theta - y \cos 3\theta = 3a \sin \theta$; hence, for the corresponding point on the reciprocal polar, $\frac{x}{\sin 3\theta} = \frac{-y}{\cos 3\theta} = \frac{a}{3 \sin \theta}$,

whence $x = a(1 - \frac{4}{3}\sin^2\theta)$, $x^2 + y^2 = \frac{a^2}{9\sin^2\theta}$, and the equation is

$$(x^2+y^2)(a-x) = \tfrac{4}{27}a^3.$$

If $x+iy = p$ be tangent from a focus, the equation

$$p(x-iy)\left(\frac{a}{p}(x+iy) - x\right) = \frac{4}{27}\frac{a^3}{p^2}(x+iy)^2$$

must have equal roots; hence

$$4\left(a-p-\frac{4}{27}\frac{a^3}{p^2}\right)\left(-a-\frac{4}{27}\frac{a^3}{p^2}\right) = \left(p-\frac{8}{27}\frac{a^3}{p^2}\right)^2,$$

or $(p-2a)^2 - \frac{16}{27}\frac{a^3}{p^2}(p+a-a+p) = 0$; or $p^2(p-2a)^2 = \frac{32a^3}{27}p$,

which has roots 0, $\frac{2}{3}a$, $\frac{2}{3}a$, $\frac{8}{3}a$. (Of course the double focus is the node, and is known beforehand as the point reciprocal to the double tangent of the cardioid $x = \frac{3}{2}a$.)

If r_1, r_2, r_3 be the three vectors for any point on the curve

$$r_1^2 = \frac{4}{27}\frac{a^3}{a-x}, \quad r_2^2 = (x-\tfrac{2}{3}a)^2 + y^2 = \frac{4}{27}\frac{a^3}{a-x} - \tfrac{4}{3}ax + \tfrac{4}{9}a^2 = \tfrac{4}{3}a\frac{(x-\frac{2}{3}a)^2}{a-x},$$

$$r_3^2 = \frac{4}{27}\frac{a^3}{a-x} - \tfrac{16}{3}ax + \tfrac{64}{9}a^2 = \frac{4}{27}\frac{a(6x-7a)^2}{a-x};$$

hence, for any point on the oval ($x < \frac{3}{2}a$), we have

$$\frac{r_1}{a} = \frac{r_2}{2a-3x} = \frac{r_3}{7a-6x} \quad \text{or} \quad 3r_1 + 2r_2 = r_3.$$

For the sinuous branch, $\frac{r_1}{a} = \frac{r_2}{3x-2a} = \frac{r_3}{7a-6x}$, and $3r_1 = 2r_2 + r_3$.

II. *Solution by* Professor TOWNSEND, M.A., F.R.S.

The cardioid being a tricuspal quartic, symmetrical with respect to an axis of figure, touching the axis cuspally at its single real point of meeting with the quiescent circle of its ordinary generation, intersecting it again orthogonally at three times the same distance in the opposite direction from the centre of the circle, passing cuspally through the two circular points at infinity, and having real double contact with a bitangent chord perpendicular to the axis, at points determining an equilateral triangle with the centre of the circle, of which its axial cusp is the centre of figure; its polar reciprocal to the quiescent circle is consequently a crunodal cubic, symmetrical with respect to the same axis of figure, intersecting the axis doubly at angles of 60° at a distance from the centre towards its axial cusp equal to two-thirds of the radius of the circle, intersecting it again singly at right angles at half the same distance in the opposite

direction from the centre, passing inflexionally through the two circular points at infinity, and having real inflexional contact with a terminal asymptote perpendicular to the axis, the tangent, viz., to the circle at its axial cusp.

Hence, denoting by a the radius of the circle, and taking for axes of x and of y respectively the axis and asymptote of the cubic, the positive direction of x being towards the centre of the circle; the equation

$$y^2x + (x-\tfrac{1}{3}a)^2(x-\tfrac{4}{3}a) = 0,$$

or its equivalent $$(x^2+y^2)x - 2ax^2 + a^2x - \tfrac{4}{27}a^3 = 0,$$

is easily seen to satisfy all the aforesaid particulars of the latter, and shows that, as stated in the question, the curve has three real axial foci situated at the distances $+a$, $+\frac{1}{3}a$, and $-\frac{4}{3}a$ from the origin, the first coinciding with the centre of the circle and the second with the node of the curve; a circle of evanescent radius round each as centre having, as may be easily seen, double contact with the curve, real in the case of the nodal and imaginary in those of the other two foci.

To find the vectorial equation of the curve referred to its three foci, or, which is the same thing, to determine the values of the three ratios $\lambda : \mu : \nu$ in the linear equation $\lambda\rho + \mu\sigma + \nu\tau = 0$, where ρ, σ, τ are the three focal vectors, in the above order of the foci, of any point on the curve. Since, for the nodal point on the axis $\lambda \pm 0\mu \cdot 3\nu = 0$, for the apsidal point on the axis $\lambda + 3\mu + 9\nu = 0$, and for the inflexional point on the asymptote $\lambda + \mu + \nu = 0$, therefore at once, as stated in the question, $\lambda : \mu : \nu \quad 3 : 2 : -1$ for the loop, and $= 3 : -2 : -1$ for the remainder of the curve.

6550. (By W. R. Westropp Roberts, M.A.)—A circular elastic lamina, of uniform thickness and isotropic material, is capable of motion round an axis passing through its centre and perpendicular to its plane; being given the strength of the material, determine the angular velocity which is just sufficient to produce permanent alteration of form.

Solution by the Proposer.

Let u, v be the projections of the displacement of any point P, on two rectangular axes in the plane of the lamina, which must evidently be of the form $$u = x \cdot f(r), \quad v = y \cdot f(r);$$

whence, with the usual notation,

$$\theta = 2f(r) + rf'(r), \quad N_1 = \lambda\theta + 2\mu\left\{f(r) + \frac{x^2}{r} \cdot f'(r)\right\}, \quad T_3 = 2\mu xyf'(r);$$

and these values, substituted in the equation

$$\frac{dN_1}{dx} + \frac{dT_3}{dy} + m\Omega^2 x = 0, \quad \text{give } (\lambda + 2\mu)\frac{d\theta}{dr} = -m\Omega^2 r;$$

consequently $$f(r) = \alpha + \frac{\beta}{r^2} - \gamma\frac{r^2}{4}, \quad \text{where } \gamma = \frac{m\Omega^2}{2(\lambda + 2\mu)}.$$

Now, we must have $\beta = 0$, since both u and v vanish when $r = 0$;

hence $$\theta = 2\alpha - \gamma r^2, \; f(r) = \alpha - \gamma \frac{r^2}{4}.$$

Following LAMÉ'S notation, R_1, the normal elastic force on an element plane perpendicular to any radius r, is found from the above value of N_1 by putting $x = r$; and Φ_2, the normal elastic force on an element plane of a section by a plane through the axis of rotation, by putting $x = 0$; hence

$$R_1 = 2\alpha(\lambda + \mu) - \gamma r^2(\lambda + \tfrac{3}{2}\mu), \quad \Phi_2 = 2\alpha(\lambda + \mu) - \gamma r^2(\lambda + \tfrac{1}{2}\mu).$$

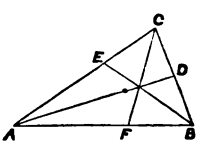
If we suppose $R_1 = 0$ when $r = a$, the radius of the plate, and A the greatest value of Φ_2, which occurs when $r = 0$, we find

$$\Omega^2 = \frac{4A}{ma^2} \cdot \frac{(\lambda + 2\mu)}{(2\lambda + 3\mu)}.$$

4188. (By Dr. HART.)—If the angles of a plane triangle be bisected by the given lines a, b, c, terminating in the opposite sides respectively, it is required to find the sides.

Solution by Prof. EVANS, M.A.; *the* Rev. U. J. KNISELY, M.A.; *and others.*

Let ABC be a triangle similar to the required triangle, $AD = a\beta$, $BE = b\beta$, $CF = c\beta$, $AB = 1$, $BC = x$, $AC = y$. Then $AF = \frac{y}{x+y}$, $FB = \frac{x}{x+y}$, $BD = \frac{x}{1+y}$, $DC = \frac{xy}{1+y}$, $CE = \frac{xy}{x+1}$, $EA = \frac{y}{x+1}$;



and since $AC \cdot BC = AF \cdot FB + CF^2$,

$$AB \cdot AC = BD \cdot DC + AD^2, \quad AB \cdot BC = AE \cdot EC + BE^2,$$

we have $CF^2 : AD^2 = AC \cdot BC - AF \cdot FB : AB \cdot AC - BD \cdot DC$,

$$CF^2 : BE^2 = AC \cdot BC - AF \cdot FB : AB \cdot BC - AE \cdot EC;$$

or $$c^2 : a^2 = xy\left\{1 - \left(\frac{1}{x+y}\right)^2\right\} : y\left\{1 - \left(\frac{x}{1+y}\right)^2\right\} \quad \ldots\ldots\ldots\ldots(1),$$

$$c^2 : b^2 = xy\left\{1 - \left(\frac{1}{x+y}\right)^2\right\} : x\left\{1 - \left(\frac{y}{x+1}\right)^2\right\} \quad \ldots\ldots\ldots\ldots(2).$$

When a, b, c are given in numbers, x and y may be found from (1) and (2); and thence the sides $\frac{1}{\beta}$, $\frac{x}{\beta}$, $\frac{y}{\beta}$ of the required triangle, since

$$\beta^2 = \frac{xy}{c^2}\left\{1 - \left(\frac{1}{x+y}\right)^2\right\}.$$

[Dr. HART remarks that he is convinced that the sides of the triangle cannot be explicitly exhibited in terms of the three given lines.]

6281. (By Prof. WOLSTENHOLME, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a parabola passing through the two given foci and having its axis along the given asymptote.

Solution by W. R. WESTROPP ROBERTS, M.A.

Let the two given foci be $(a, 0), (-a, 0)$, and the given asymptote $px+qy=1$; and let the vector equation of the cubic for these two foci and a third (for the infinite branch) be $lr_1+mr_2=(l+m)r_3$, where

$$r_1=\{(x-a)^2+y^2\}^{\frac{1}{2}},\quad r_2=\{(x+a)^2+y^2)\}^{\frac{1}{2}},\quad r_3=\{(x-X)^2+(y-Y)^2\}^{\frac{1}{2}},$$

then
$$r_1=(x^2+y^2)^{\frac{1}{2}}\left\{1-\frac{2ax-a^2}{2(x^2+y^2)}-\frac{4a^2x^2}{8(x^2+y^2)^2}+\dots\right\},$$

$$r_2=(x^2+y^2)^{\frac{1}{2}}\left\{1+\frac{2ax+a^2}{2(x^2+y^2)}-\frac{4a^2x^2}{8(x^2+y^2)^2}+\dots\right\},$$

$$r_3=(x^2+y^2)^{\frac{1}{2}}\left\{1-\frac{2(xX+yY)-X^2-Y^2}{2(x^2+y^2)}-\frac{4(xX+yY)^2}{8(x^2+y^2)^2}\right\};$$

and we shall have for the asymptote

$$\left(2ax-a^2+\frac{a^2q^2}{p^2+q^2}\right)-m\left(2ax+a^2-\frac{a^2q^2}{p^2+q^2}\right)$$
$$=(l+m)\left\{2xX+2yY-X^2-Y^2+\frac{(qX-pY)^2}{p^2+q^2}\right\};$$

and, making this coincide with $px+qy=1$, we get

$$(l+m)\frac{2Y}{q}=(l+m)\left\{X^2+Y^2-\frac{(qX-pY)^2}{p^2+q^2}-a^2+\frac{a^2q^2}{p^2+q^2}\right\},$$

or
$$\frac{2(p^2+q^2)}{q}Y=(pX+qY)^2-p^2a^2;\ \text{therefore \&c.}$$

[One significant term was omitted, in each expansion, in the Solution given of this Question on p. 115 of Vol. XXXIV. of the *Reprint*.]

5344. (By S. TEBAY, B.A.)—n grocers have $a_1, a_2, \dots a_n$ lbs. of tea respectively ($a_1 < a_2 <$ &c.); all sell a certain number of lbs. at the same price, and afterwards the remainders at a different price, and all realize equal sums from the two sales. Show how this can be done, and deduce a simple particular solution.

Solution by Prof. EVANS, M.A.; *the* PROPOSER; *and others.*

Let P be the price at which $x_1, x_2, \dots x_n$ lbs. are sold, and P′ the price at which the remainders are sold. Then we have

$$Px_1+P'(a_1-x_1)=Px_2+P'(a_2-x_2)=Px_3+P'(a_3-x_3)=\text{\&c.}$$

Therefore $\frac{P}{P'} = 1-\frac{a_2-a_1}{x_2-x_1} = 1-\frac{a_3-a_2}{x_3-x_2} = \&c.;$

whence $x_1(a_3-a_2)-x_2(a_3-a_1)+x_3(a_2-a_1) = 0,$

$x_2(a_4-a_3)-x_3(a_4-a_2)+x_4(a_3-a_2) = 0, \&c.$

There are $n-2$ equations, so that, if x_1, x_2 be assumed, the others follow.

Let a_1, a_2, &c. be in arithmetical progression, d the common difference; then $x_1-2x_2+x_3 = 0$, $x_2-2x_3+x_4 = 0$, &c.; so that x_1, x_2, &c. are in arithmetical progression. Let d' be the difference, then $\frac{P}{P'} = 1-\frac{d}{d'}$. This being reduced or modified, the numerator can be taken for P, and the denominator for P'; the limiting inequality for d' being

$$x_1+(n-1)d' < a_n.$$

Let $a_1 = 10$, $a_2 = 13$, $a_3 = 16$, $a_4 = 19$; so that $d = 3$; and take $x_1 = 2$. Then $2+3d' < 19$, or $d' < 5\frac{2}{3}$; therefore $d' = 4$ or 5. Let $\frac{P}{P'} = 1-\frac{3}{5} = \frac{2}{5}$; and take $P = 2s.$, $P' = 5s.$ The number of pounds at $2s.$ = 2, 7, 12, 17; and the number at $5s.$ = 8, 6, 4, 2; and each realizes 44 shillings.

5121. (By S. Tebay, B.A.)—If n letters be formed in groups of p letters in each, no q letters being repeated, but every q of the n letters exhibited; then, on erasing x letters, shew, x being less than q, that the number of groups containing r of the x letters is

$$N_r = \frac{x!}{r!\,(x-r)!}\cdot\frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-r)!}\cdot\frac{(n-p)!}{(n-p-x+r)!}.$$

Solution by Belle Easton; *the* Proposer; *and others.*

To find N_x, compare the combinations in q, each containing the x letters; thus, $\frac{(p-x)!}{(q-x)!\,(p-q)!}N_x = \frac{(n-x)!}{(q-x)!\,(n-q)!}$, and $N_x = \frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x)!}$.

To find N_{x-1}, compare the combinations in q, each containing $x-1$ of the x letters; thus,

$$\frac{[p-(x-1)]!}{[q-(x-1)]!\,(p-q)!}N_{x-1}+x\frac{(p-x)!}{[q-(x-1)]!\,(p-q-1)!}N_x = x\frac{(n-x)!}{[q-(x-1)]!\,(n-q-1)!};$$

therefore $N_{x-1} = x\frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x+1)!}(n-p).$

To find N_{x-2}, compare the combinations in q, each containing $x-2$ of the x letters; thus,

$$\frac{\lceil p-(x-2)\rceil !}{[q-(x-2)]\,!\,(p-q)!}\,N_{x-2}+(x-1)\,\frac{\lceil p-(x-1)\rceil !}{[q-(x-2)]\,!\,(p-q-1)\,!}\,N_{x-1}$$
$$+\frac{x(x-1)}{1.2}\cdot\frac{(p-x)\,!}{[q-(x-2)]\,!\,(p-q-2)\,!}\,N_x$$
$$=\frac{x(x-1)}{1.2}\cdot\frac{(n-x)\,!}{[q-(x-2)]\,!\,(n-q-2)!};$$

therefore $N_{x-2}=\frac{x(x-1)}{1.2}\cdot\frac{(n-x)\,!\,(p-q)\,!}{(n-q)\,!\,(p-x+2)\,!}\,(n-p)\,(n-p-1).$

And so on. Thus we find

$$N_{x-r}=\frac{x\,!}{r\,!\,(x-r)\,!}\cdot\frac{(n-x)\,!\,(p-q)\,!}{(n-q)\,!\,(p-x+r)\,!}\cdot\frac{(n-p)\,!}{(n-p+r)\,!};$$

or, writing r for $x-r$,

$$N_r=\frac{x\,!}{r\,!\,(x-r)\,!}\cdot\frac{(n-x)\,!\,(p-q)\,!}{(n-q)\,!\,(p-r)\,!}\cdot\frac{(n-p)\,!}{(n-p-x+r)\,!}.$$

If u_n denote the whole number of combinations,

$$u_{n-x}=u_n-(N_1+N_2+\ldots+N_x).$$

5846. (By W. H. H. Hudson, M.A.)—If the normal at a point P of the lemniscate meet the axis in G, and if GQ, CR be drawn perpendicular to CG, CP respectively to meet the circle described on the line joining the poles of the lemniscate as diameter in Q, R; prove that

$$GQ : CQ = CQ^2 : RP^2.$$

Solution by J. A. Kealy, M.A.; J. O'Regan; *and others.*

Let the equation of lemniscate be $(x^2+y^2)^2=2a^2(x^2-y^2)$, where $2a$ is the distance between the poles, and let P be the point (x, y), then we have $CQ=a$, $RP^2=a^2+CP^2=a^2+x_1^2+y_1^2$;

Normal at P, $y-y_1=\frac{y_1^3+x_1^2y_1+a^2y_1}{x_1^3+x_1y_1^2-a^2x_1}(x-x')$, $CG=\frac{2a^2x_1}{a^2+x_1^2+y_1^2}$,

$$GQ=\frac{a^3}{a^2+x_1^2+y_1^2},\quad GQ : CQ=a^2 : a^2+x_1^2+y_1^2=CQ^2 : RP^2.$$

6595. (By the late S. M. Drach, F.R.A.S.)—

If six-digit number N be so reckoned
That twice (6th less 3rd) plus thrice (5th less 2nd),
With 4th less 1st, announce sum sevenfold,
Whole number N by *sevens* can be told.

Quadruple (6th less 3rd) plus thrice (5th less 2nd)
With 1st less 4th, is test if *thirteen*'s beckoned.

Solution by W. B. Grove, B.A.; Prof. Droz, M.A.; *and others.*

Let $$N \equiv 10^5 r_6 + 10^4 r_5 + 10^3 r_4 + 10^2 r_3 + 10 r_2 + r_1$$
$$= 7ar_6 - 2r_6 + 7br_5 - 3r_5 + 7cr_4 - r_4 + 7a'r_3 + 2r_3 + 7b'r_2 + 3r_2 + r_1$$
$$= 7M - \{2(r_6 - r_3) + 3(r_5 - r_2) + (r_4 - r_1)\}.$$
Also $$N = 13ar_6 + 4r_6 + 13br_5 + 3r_5 + 13cr_4 - r_4 + 13a'r_3 - 4r_3 + 13b'r_5 - 3r_2 + r_1$$
$$= 13M + 4(r_6 - r_3) + 3(r_5 - r_2) + (r_1 - r_4).$$
The process can be similarly extended to any number of digits.

6552. (By J. J. Walker, M.A.)—If $u = x^n \log x$, prove that there is a form of $\frac{d^r u}{dx^r}$ other than that given by Leibnitz's Theorem; viz.,
$$n \,.\, n-1 \,\ldots\, n-r+1 \,.\, x^{n-r}\left(\log x + \frac{1}{n-r+1} + \frac{1}{n-r+2} \ldots + \frac{1}{n}\right).$$

Solution by the Proposer.

If $u = x^n \log x$, we have $\frac{1}{n}\frac{du}{dx} = x^{n-1}\log x + x^{n-1}$,

$$\frac{1}{n \,.\, n-1}\frac{du^2}{dx^2} = x^{n-2}\log x + \frac{x^{n-2}}{n-1} + \frac{x^{n-2}}{n};$$

and generally $\frac{1}{n \ldots (n-r+1)}\frac{d^r u}{dx^r} = x^{n-r}\log x + \frac{x^{n-r}}{n-r+1} + \frac{x^{n-r}}{n-r+2} \ldots + \frac{x^{n-r}}{n}$.

For, differentiating again, and dividing by $n-r$,
$$\frac{1}{n \ldots (n-r)}\frac{d^{r+1}u}{dx^{r+1}} = x^{n-r-1}\log x + \frac{x^{n-r-1}}{n-r} + \frac{x^{n-r-1}}{n-r+1} + \ldots + \frac{x^{n-r-1}}{n}.$$
Thus, if the formula holds for one value of r, it holds for the next integer; but it holds for $r = 1$, $r = 2$.

4185. (Proposed by W. H. H. Hudson, M.A.)—A series of ellipsoids are constructed, each of which is the ellipsoid of gyration of the preceding one about its centre; find the form of the nth ellipsoid so constructed, the first being given, and all the ellipsoids being supposed of uniform density: prove also that the ultimate form of the nth ellipsoid, when n is increased indefinitely, is that of a sphere.

Solution by J. L. KITCHIN, M.A.; A. MARTIN, M.A.; *and others.*

Let the original ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$;

then, putting $d^2=a^2+b^2+c^2$, the successive ellipsoids of gyration will be

$$\frac{x^2}{\frac{b^2+c^2}{5}}+\frac{y^2}{\frac{a^2+c^2}{5}}+\frac{z^2}{\frac{a^2+b^2}{5}}=1, \qquad \frac{x^2}{\frac{d^2+a^2}{5^2}}+\frac{y^2}{\frac{d^2+b^2}{5^2}}+\frac{z^2}{\frac{d^2+c^2}{5^2}}=1,$$

$$\frac{x^2}{\frac{3d^2-a^2}{5^3}}+\frac{y^2}{\frac{3d^2-b^2}{5^3}}+\frac{z^2}{\frac{3d^2-c^2}{5^3}}=1, \quad \&c., \&c.$$

Now, when n is very large, $\frac{a^2}{5^n}$, &c. are very small; hence

$$\frac{x^2}{\frac{ma^2}{5^n}}+\&c.\ldots\ldots=1, \quad \text{therefore } x^2+y^2+z^2=\frac{ma^2}{5^n}, \text{ a sphere.}$$

5194. (By A. W. PANTON, M.A.)—If the coordinates (x, y, z) of a point on a unicursal cubic be proportional to $a_1\theta^3+b_1\theta^2+c_1\theta+d_1$, $a_2\theta^3+b_2\theta^2+c_2\theta+d_2$, $a_3\theta^3+b_3\theta^2+c_3\theta+d_3$; show that the three values of θ at the points of inflexion are the roots of the cubic equation

$$\begin{vmatrix} 1 & -3\theta & 3\theta^2 & -\theta^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

Solution by Prof. NASH, M.A.; Prof. MATZ, M.A.; *and others.*

Any straight line $\lambda x+\mu y+\nu z=0$ meets the cubic in three points given by the equation

$$(\lambda a_1+\mu a_2+\nu a_3)\,\theta^3+(\lambda b_1+\mu b_2+\nu b_3)\,\theta^2+(\lambda c_1+\mu c_2+\nu c_3)\,\theta + (\lambda d_1+\mu d_2+\nu d_3)=0 \ldots\ldots(1),$$

and if this line be the tangent at the point of inflexion $\theta=\alpha$, this equation must reduce to $k(\theta-\alpha)^3=0$; whence α is given by the determinant equation obtained by eliminating λ, μ, ν, k from the equations to the left;

therefore $\begin{vmatrix} \lambda a_1+\mu a_2+\nu a_3-k=0 \\ \lambda b_1+\mu b_2+\nu b_3+3k\alpha=0 \\ \lambda c_1+\mu c_2+\nu c_3+3k\alpha^2=0 \\ \lambda d_1+\mu d_2+\nu d_3+k\alpha^3=0 \end{vmatrix}$, $\begin{vmatrix} 1 & -3\alpha & 3\alpha^2 & -\alpha^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$

Using Prof. WOLSTENHOLME's notation, the equation is

$$D\alpha^3+3C\alpha^2+3B\alpha+A=0 \ldots\ldots(2).$$

If $\lambda x+\mu y+\nu z=0$ be the axis of inflexion, the equations (1) and (2) must have the same roots. Therefore

$$\frac{\lambda a_1+\mu a_2+\nu a_3}{D}=\frac{\lambda b_1+\mu b_2+\nu b_3}{3C}=\frac{\lambda c_1+\mu c_2+\nu c_3}{3B}=\frac{\lambda d_1+\mu d_2+\nu d_3}{A}.$$

These three equations are not independent, but, solving the first two, we obtain $\dfrac{\lambda}{(b_2c_3)+3(d_2a_3)} = \dfrac{\mu}{(b_3c_1)+3(d_3a_1)} = \dfrac{\nu}{(b_1c_2)+3(d_1a_2)}$.

For a unicursal quartic given by the equations

$$x = a_1\theta^4 + b_1\theta^3 + c_1\theta^2 + d_1\theta + e_1,\quad y = a_2\theta^4 + b_2\theta^3 + c_2\theta^2 + d_2\theta + e_2,$$
$$z = a_3\theta^4 + b_3\theta^3 + c_3\theta^2 + d_3\theta + e_3;$$

the equation giving the values of θ at the points of inflexion is

$$\begin{vmatrix} 6\theta^2a_1 + 3\theta b_1 + c_1, & 3\theta^2b_1 + 4\theta c_1 + 3d_1, & \theta^2c_1 + 3\theta d_1 + 6e_1 \\ 6\theta^2a_2 + 3\theta b_2 + c_2, & 3\theta^2b_2 + 4\theta c_2 + 3d_2, & \theta^2c_2 + 3\theta d_2 + 6e_2 \\ 6\theta^2a_3 + 3\theta b_3 + c_3, & 3\theta^2b_3 + 4\theta c_3 + 3d_3, & \theta^2c_3 + 3\theta d_3 + 6e_3 \end{vmatrix} = 0,$$

or
$$2\theta^6(a_1b_2c_3) + 6\theta^5(a_1b_2d_3) + \theta^4\{12(a_1b_2e_3) + 6(a_1c_2d_3)\}$$
$$+ \theta^3\{16(a_1c_2e_3) + 3(b_1c_2d_3)\} + \theta^2\{6(b_1c_2e_3) + 12(a_1d_2e_3)\}$$
$$+ 6\theta(b_1d_2e_3) + 2(c_1d_2e_3) = 0;$$

where $(a_1b_2c_3)$ denotes the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

6586. (By D. Edwardes.)—A string of length $2l$ is suspended from two points in the same horizontal line distant $2a$ from each other. If the distance between the points be slightly increased by the quantity 2δ, prove that the vertex of the catenary will ascend through the space $\dfrac{z^2+cz-l^2}{cl-az-ac}\cdot\delta$, where c is tension at lowest point, and z the depth of lowest point below the line of suspension.

Solution by the Rev. T. R. Terry, M.A.; Lizzie A. Kittudge; *and others.*

With the given notation, $l = \frac{1}{2}c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)$, $l^2 + c^2 = (z+c)^2$.....(1, 2), where l remains constant, but z and c vary with a.

From (1), $0 = \frac{1}{2}c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)\dfrac{dc}{da} - \frac{1}{2}a\left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right)\dfrac{dc}{da} + \frac{1}{2}c\left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right)$,

or
$$\frac{dc}{da} = \frac{-(z+c)c}{cl-az-ac}.$$

From (2), $c = (z+c)\left(\dfrac{dz}{dc}+1\right)$ or $\dfrac{dz}{dc} = -\dfrac{z}{z+c}$;

therefore $\dfrac{dz}{da} = \dfrac{zc}{cl-az-ac} = -\dfrac{z^2+zc-l^2}{cl\quad az-ac}$, which is the required result.

[The string is supposed uniform, and c is the length of the string whose weight is equal to the tension at the lowest point.]

6058. (By F. Morley, B.A.)—Two equal perfectly elastic particles fall simultaneously from points in the same vertical line at heights h, h' from a fixed perfectly elastic plane. Show that the height of the point of their first collision from the plane is, x being integral,

$$\frac{[(2x+1)^2h'-h][h-(2x-1)^2h']}{16x^2h'}, \text{ where } h>(2x-1)^2h',\ <(2x+1)^2h'.$$

Solution by the Rev. T. R. Terry, M.A.; J. O'Regan; *and others.*

Suppose the lower ball strike the plane x times before collision. Time elapsed before this x^{th} impact $=(2x-1)\left(\frac{2h'}{g}\right)^{\frac{1}{2}}$. At this instant velocity of upper ball $=(2x-1)(2gh')^{\frac{1}{2}}$, and distance travelled $=(2x-1)^2h'$.

Hence, condition that collision takes place between x^{th} and $(x+1)^{\text{th}}$ impact is $h>(2x-1)^2h'$ and $<(2x+1)^2h'$. If T be the time between x^{th} impact and the collision, we have $T=\frac{h-(2x-1)^2h'}{2x(2gh')^{\frac{1}{2}}}$; therefore height above plane $=T(2gh')^{\frac{1}{2}}-\frac{1}{2}gT^2=$ &c., as in the Question.

6590. (By E. B. Elliott, M.A.)—If there be any distribution of mass M in space, if I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, and A and B respectively; prove that the moment of inertia about any fourth parallel line may be expressed in the form

$$xI_a+yI_b+zI_c-M(a^2yz+b^2zx+c^2xy),$$

x, y, z being coordinates whose sum is unity of the fourth line with regard to the prism formed by the three.

Solution by J. W. Russell, M.A.; *the* Rev. T. R. Terry, M.A.; *and others.*

Let a plane through O, the centre of gravity of the mass, cut at right angles the three given lines in ABC, and the fourth line in P. Then, if (xyz) be the areal coordinates of P, and $(x'y'z')$ those of O referred to the triangle ABC, and I, I_0 the moments of inertia about parallel axes through P and O respectively, we have

$$\begin{aligned}
I &= I_0-M\{a(y-y')(z-z')+b^2(z-z')(x-x')+c'(x-x')(y-y')\},\\
I_a &= I_0-M\{a^2y'z' \qquad +b^2(-z')(1-x')+c^2(-x')(1-y')\},\\
I_b &= I_0-M\{a^2(1-y')(-z') \ +b^2z'x' \qquad +c^2(-x')(1-y')\},\\
I_c &= I_0-M\{a^2(-y')(1-z') \ +b^2(1-z')(-x') \ +c^2x'y'\};
\end{aligned}$$

whence, obviously, $I=xI_a+yI_b+zI_c-M(a^2yz+b^2zx+c^2xy)$.

6325, 6367, 6501, 6562. (By R. KNOWLES, B.A., L.C.P.; Rev. D. THOMAS, M.A.; and others.)—In Quest. 6292, prove that

$$p_1 - 2p_2 + \ldots + (n-1)(-)^{n-2}p_{n-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\tfrac{1}{2}p_1 - \tfrac{1}{3}p_2 + \ldots + \frac{1}{n+1}(-)^{n-1}p_n = \frac{n}{n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\tfrac{1}{3}p_1 - \tfrac{1}{4}p_2 + \ldots + \frac{1}{n+2}(-1)^{n-1}p_n = \frac{n(n+3)}{2(n+1)(n+2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

$$1.2p_1 - 2.3p_2 + \ldots + (n-1)n(-1)^{n-2}p_{n-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

$$\frac{p_1}{2.3} - \frac{p_2}{3.4} + \ldots + \frac{1}{(n+1)(n+2)}(-1)^{n-1}p_n = \frac{n}{2(n+2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5);$$

also, if p_r denote the coefficient x^r in the expansion of $(1+x)^n$, where n is a positive integer; and if m be also any positive integer, then

$$\frac{p_0}{m+1} - \frac{p_1}{m+2} + \ldots + \frac{(-1)^n p_n}{m+n+1} = \frac{n(n-1)(n-2)\ldots 1}{(m+1)(m+2)\ldots(m+n+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Solution by J. J. WALKER, M.A.

Considering the coefficient of x^{n+m+1} in the development of

$$u = (1+x)^n \log_e(1+x) = (p_0x^n + p_1x^{n-1} + \ldots + p_n)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right),$$

which is $(-)^m \mathrm{F}(m+1) = (-1)^m\left(\frac{p_0}{m+1} - \frac{p_1}{m+2} \ldots \frac{(-)^n p_n}{m+n+1}\right)$,

and equating it with

$$\frac{1}{1.2\ldots n+m+1}\frac{d^{n+m+1}u}{dx^{n+m+1}}, \quad \text{or} \quad \frac{(-1)^m}{1.2\ldots(n+m+1)} \quad \frac{n.n-1\ldots1.1.2\ldots m}{(1+x)^{m+1}},$$

after making $x = 0$, the equality in 6562 is proved; and the first equality in 6325 is the particular case of the above, in which $m = 0$, since $\frac{n}{n+1} = p_0 - \frac{1}{n+1}$; that in 6367 is the case of $m = 1$.

Again, $\mathrm{F}(m) - \mathrm{F}(m+1)$ gives

$$\frac{p_0}{m(m+1)} - \ldots + \ldots (-1)^n \frac{p_n}{(m+n)(m+n+1)} = \frac{(n+1)n\ldots 2.1}{m(m+1)\ldots(m+n+1)},$$

which is a generalization of the second equality in 6501, the latter being the particular case of $m = 1$, since $\frac{n}{2(n+2)} = \frac{p_0}{1.2} - \frac{1}{n+2}$.

To verify the first equality of Quest. 6325, consider the coefficient of $-x^{n-1}$ in the development of $u^n\frac{du^{-1}}{dx}$, where $u = 1+x$, that is, in

$$(x^n + p_1x^{n-1} + \ldots)(-1 + 2x - 3x^2 + \ldots).$$

It is $p_1 - 2p_2 + 3p_3 - \ldots$. But $u^n\frac{du^{-}}{dx}$ also $= -(1+x)^{n-2}$, containing no power of x higher than $n-2$.

Again, consider $u^n \frac{d^2u^{-1}}{dx^2}$. The coefficient of x^{n-1} in the form

$$(x^n + p_1 x^{n-1} + \ldots)(1 \cdot 2 - 2 \cdot 3x + 3 \cdot 4x^2 - 4 \cdot 5x^3 + \ldots)$$

is the series of the first equality in Quest. 6501. But $u^n \frac{d^2u^{-1}}{dx^2}$ also equals $2(1+x)^{n-3}$, and this contains no higher power of x than x^{n-3}.

By considering, similarly, $u^n \frac{d^ru^{-1}}{dx^r}$, where r is any integer less than n, we obtain the more general results

$$1 \cdot 2 \ldots rp_1 - 2 \cdot 3 \ldots (r+1)p_2 + \ldots\ldots (-1)^n n(n+1) \ldots\ldots (n+r-1)p_n = 0,$$
$$1 \cdot 2 \ldots rp_2 - 2 \cdot 3 \ldots (r+1)p_3 + \ldots (-1)^{n-1}(n-1)n \ldots (n+r-2)p_{n-1} = 0,$$

and so on.

6579. (By J. J. Walker, M.A.)—If P, P′ are two points on an ellipse, the foci of which are S, H; and if X, X′, Y are any points in the productions of SP, SP′, PP′ respectively; prove that

$$\cos\tfrac{1}{2}(\mathrm{XPY} + \mathrm{X'P'Y}) : \cos\tfrac{1}{2}(\mathrm{HPY} + \mathrm{HP'Y}) = \cos\tfrac{1}{2}\mathrm{PSP'} : \cos\tfrac{1}{2}\mathrm{PHP'}.$$

Solution by G. Turriff, M.A.; *the* Rev. T. R. Terry, M.A.; *and others.*

Since
$$\frac{\mathrm{SP'} - \mathrm{SP}}{\mathrm{PP'}} = \frac{\mathrm{HP} - \mathrm{HP'}}{\mathrm{PP'}},$$

$$\frac{\sin \mathrm{XPY} - \sin \mathrm{X'P'Y}}{\sin \mathrm{PSP'}} = \frac{\sin \mathrm{HP'Y} - \sin \mathrm{HPY}}{\sin \mathrm{PHP'}};$$

therefore
$$\frac{\cos\frac{1}{2}(\mathrm{XPY} + \mathrm{X'P'Y})}{\cos\frac{1}{2}\mathrm{PSP'}} = \frac{\cos\frac{1}{2}(\mathrm{HPY} + \mathrm{HP'Y})}{\cos\frac{1}{2}\mathrm{PHP'}}.$$

[In the limit, when P′ coincides with P, this gives $\cos \mathrm{XPY} = \cos \mathrm{HPY}$; therefore $\angle \mathrm{XPY} = \mathrm{HPY}$, a well-known result. The same proof establishes the property for the hyperbola, by interchanging HP, HP′, and writing S for X.]

5433. (By Prof. Pratt, M.A.)—Given $u = \mathrm{F}(y)$. $y = \mathrm{F'}\{z + xvf(y)\}$, $v = \mathrm{F}_1 t$, $t = \mathrm{F}_2\{z + xvf_1(t)\}$; expand u in a series of ascending, positive, integral powers of x (x not being a function of z).

Solution by J. J. Walker, M.A.

As F′ does not appear to be used in the sense for which it is very convenient to reserve it, viz., the first derived of F, I will substitute F_0 for F′. Then, v being a function of x and z,

$$\frac{dy}{dx} = F'_0\{z+\ldots\}\left\{vf(y)+x\frac{dv}{dx}f(y)+xvf'(y)\frac{dy}{dx}\right\},$$

whence

$$\frac{dy}{dx} = \left[\left(v+x\frac{dv}{dx}\right)f(y)F'_0\{z+xvf(y)\}\right] : \left[1-xvf'(y)F'_0\{z+\ldots\}\right].$$

Again, $\frac{dy}{dz} = F'_0\{z+\ldots\}\left\{1+x\frac{dv}{dz}f(y)+xvf'(y)\frac{dy}{dz}\right\}$, whence

$$\frac{dy}{dz} = \left[\left\{1+x\frac{dv}{dz}f(y)\right\}F'_0\{z+\ldots\}\right] : \left[1-xvf'(y)F'_0\{z+\ldots\}\right];$$

so that $\frac{dy}{dx} = \left[\left(v+x\frac{dv}{dx}\right)f(y) : \left\{1+x\frac{dv}{dz}f(y)\right\}\right]\frac{dy}{dz}$.

Following the steps of the proof of LAGRANGE's theorem, the general formula $\frac{d^n u}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left[\left\{\left(v+x\frac{dv}{dx}\right)f(y)\right\}^n\frac{du}{dz} : \left\{1+x\frac{dv}{dz}f(y)\right\}^n\right]$ is obtained; so that, when $x = 0$,

$$\frac{d^n u}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left[\{vf(y)\}^n\frac{dF\{F_0(z)\}}{dz}\right],$$

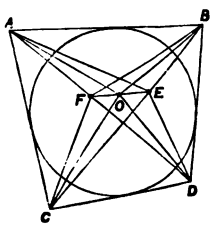
wherein $y = F_0(z)$, $v = F_1\{F_2(z)\}$; and the required expansion is, restoring F' in place of F_0,

$$u = F\{F'(z)\}+xF_1\{F_2(z)\}f\{F'(z)\}\frac{dF\{F'(z)\}}{dz}+\frac{x^2}{1.2}\frac{d}{dz}$$
$$\times\left[(F_1\{F_2(z)\}f\{F'(z)\})^2\frac{dF\{F'(z)\}}{dz}\right]$$
$$+\frac{x^3}{1.2.3}\frac{d^2}{dz^2}\left[(\ldots)^3\frac{dF\{F'(z)\}}{dz}\right]+\ldots.$$

6436. (By D. EDWARDES)—If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle lie in a straight line.

Solution by H. MURPHY; J. O'REGAN; *and others.*

Let E, F be the middles of the diagonals, and O the centre; then $\triangle AFC$ and $\triangle BFD = \frac{1}{4}$ quad. $ABDC = \triangle AEC + \triangle BED = \triangle AOC + \triangle BOD$; hence we have the given bases AC and BD, and the sum of the areas of two triangles having a common vertex; thus the locus of vertices is a straight line EOF.



[This question is a particular case of CHASLES' theorem, that the locus of the centres of conics touching the same four straight lines is a straight line.]

6596. (By Professor SYLVESTER, F.R.S.)—APB is a semi-ellipse of which S is the focus, and AB the major axis; and AMB is a semicircle drawn on AB; show (1) that if two bodies, attracted each towards its own centre of force at S, and leaving the point A at the same moment, move in these two curves, the one will be at P when the other is at M, PM being any ordinate perpendicular to AB; and (2) state also the theorem or the case when S coincides with A.

Solution by J. J. WALKER, M.A.

1. What is stated in the Question will be the case if a certain relation hold between the elliptic and circular forces, which may be most neatly expressed as that the accelerations towards S at A must be equal one to the other. For the elliptic movement, if f_1 is the acceleration at A, then, generally, at P, $f = \frac{f_1 \cdot SA^2}{SP^2} = \frac{ah^2}{b^2 SP^2}$; $\therefore f_1 = \frac{ah^2}{b^2 . SA^2}$.

Again, for the circular movement, $f' = \frac{2h'^2 AB}{MM'^3 . SM^2}$, MM' being the chord through S; and, at A, $f_1' = \frac{h'^2}{a . SA^2}$. Hence $\frac{f_1}{f_1'} = \frac{a^2h^2}{b^2h'^2}$; and, for the condition of the Quest., obviously, $h^2 : h'^2 = b^2 : a^2$, so that $f_1 = f_1'$.

2. The relation between the forces, in this case, may be stated by the condition that the accelerations at B towards A must be equal. For, generally, in the elliptic movement, $f = \frac{h^2 CG^3}{a^2b^2 AP^3}$, where CG is the length of a line drawn from the centre (C) to the tangent at P, parallel to SP, so that, at B, $f_2 = \frac{h^2}{4ab^2}$. In the circle, $f' = \frac{2h'^2 (2a)^2}{AM^5}$ generally; and, in particular, at B, $f_2' = \frac{h'^2}{4a^3}$, so that $\frac{f_2}{f_2'} = \frac{a^2h^2}{b^2h'^2}$, and therefore $f_2 = f_2'$. I understand the centre of force to be removed to A, the ellipse remaining unchanged, or not.

[Prof. SYLVESTER remarks that one force varies as d^{-2} and the other according to the law given by NEWTON, viz., $d^{-2} q^{-3}$ (q being the complete chord of which d is a part), but these limits need not be stated in the Question. When S coincides with A, the ellipse becomes a straight line, and B becomes the *starting* point of the body moving in this line, acted on by a force varying as d^{-2}; also the force in the circle then varies as d^{-5}.]

6581. (By W. R. WESTROPP ROBERTS, M.A.)—Through a fixed chord AB of a twisted cubic are drawn two planes harmonically conjugate with two fixed planes through the same chord. Show that, if the variable planes meet the cubic in two points P and Q, the chord PQ meets two fixed lines.

Solution by Professor TOWNSEND, M.A., F.R.S.

Denoting by U and V the two quadric cones, having A and B for vertices which intersect in the cubic, and in the line AB; by E and F the

two thirds points of intersection with the curve of the two fixed planes through AB, and by M and N the two unique lines of intersection of the two pairs of tangent planes to U and V along their two pairs of generating lines AE and AF, BE and BF, respectively; then, the two variable planes APB and AQB dividing harmonically in every position the angle between the two fixed planes AEB and AFB, and the two variable planes PAQ and PBQ passing consequently in every position, the former through the fixed line M and the latter through the fixed line N, the variable line PQ common to the two latter planes passes consequently in every position through the two fixed lines M and N; and therefore, &c.

If the two variable planes APB and AQB, in place of dividing harmonically in every position the angle between the two fixed planes AEB and AFB, divided it more generally in any constant anharmonic ratio; since then the two variable planes PAQ and PBQ, instead of passing respectively through the two fixed lines M and N, would touch respectively two fixed quadric cones U′ and V′ having double contact with U and V along the two pairs of lines AE and AF, BE and BF, respectively, and breaking up into the two pairs of planes touching them along those lines in the case of harmonic section, the variable line PQ, instead of passing in every position through the two fixed lines M and N, would touch in every position the two fixed cones U′ and V′, and would pass through M and N in the case only of harmonic section when the two cones break up into pairs of planes intersecting in those lines.

That, in the same case, the variable chord PQ generates a ruled quadric, passing through the cubic and through the two lines M and N, may be readily shown as follows. The ruled quadric determined by any three positions P_1Q_1, P_2Q_2, P_3Q_3 of PQ, passing through the two lines M and N, and therefore through the two points A and B, intersects consequently the cubic in eight points P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, A and B, and therefore contains it altogether; and every generator of it L of the system opposite to M and N, intersecting in consequence the curve in a pair of points P and Q connecting through M and N, coincides accordingly with a position of PQ; and therefore, &c.

When the two fixed planes AEB and AFB divide, as they may, harmonically the angle between the two planes through AB which touch the curve at A and B, then is AB a position of PQ, but otherwise not. In that case, therefore, the quadric generated by PQ passes through the complete intersection of the two cones U and V, while in all other cases it passes only through the curve, but not through the line common to them both.

6526. (By the Editor.)—From the points in which any two conjugate diameters of an ellipse meet a fixed tangent, tangents are drawn to a fixed confocal; prove that the locus of their intersection is a circle whose centre is on the normal to the fixed tangent.

Solution by C. Taylor, M.A.

From the points in which any two parallel tangents or conjugate diameters of an ellipse meet the tangent at a fixed point O, draw four tangents to a fixed confocal. In any assumed position the other four inter-

sections of these tangents lie in a circle. Consider this circle fixed, and let another quadrilateral be inscribed in it, so as to envelop the inner confocal. The opposite sides of this quadrilateral intersect at two points on the tangent at O; and it is evident that the second tangents from these two points to the original ellipse are parallel, and conversely. Hence the required locus is a circle. Making one of the parallel tangents coincide with the tangent at O, we see that the centre of the circle lies on the normal at O. [See also Question 2911, whereof an algebraical proof is given on pp. 31—33 of Vol. XIII. of the *Reprint*.]

NOTE ON PROF. CROFTON'S QUESTION 4795. *By* PROF. NASH, M.A.

The solution of this Question suggests the following query:—The theorem is its own inverse with respect to each of the foci, the constant of inversion being so chosen that the other foci are inverse points. Is this a sufficient proof of the truth of the theorem? I have frequently found that theorems repeat themselves in this way when inverted or reciprocated. For example, Euclid VI. D is its own inverse with respect to any point.

As examples in reciprocation, Quest. 1471 in Prof. WOLSTENHOLME'S *Book of Problems* may be taken, and also the theorem that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. In both of these cases only one point can be taken as origin of reciprocation, which is not generally the case with inversion.

6606. (By ELIZABETH BLACKWOOD.)—In the integral

$$\int dw \int dx \int dy \int dz\ \phi(w, x, y, z),$$

the variables are each between the limits a and $-a$, and the integration is further restricted by the condition that $(wx-yz)$ is positive; determine the limits of integration.

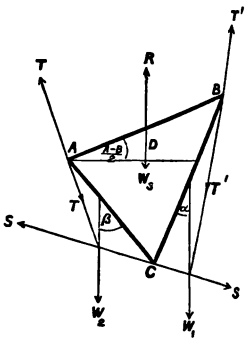
Solution by W. B. GROVE, B.A.

Let $Q \equiv w_{1'.2}\, x_{1'.2}\, y_{1'.2}\, z_{1'.2}\, p\,(wx-yz)$, where the symbols have the meanings attached to them in Mr. MCCOLL'S first paper in *Proc. Lond. Math. Soc.* Then it will be found that

$$Q = (w_{1'.0}\, x_{1'.0} + w_{0'.2}\, x_{0'.2})(y_{1'.3}\, z_{3'.2} + y_{3'.4}\, z_{1'.2} + y_{4'.2}\, z_{1'.3}) + (w_{1'.0}\, x_{0'.2} + w_{0'.2}\, x_{1'.0})\,(y_{1'.4}\, z_{3'.2} + y_{3'.2}\, z_{1'.3}).$$

Solution by D. EDWARDES; Prof. DROZ, M.A.; *and others.*

(1). Supposing the bars to be of the same thickness and density; then, since the centroid of the perimeter of a triangle is the centre of the circle inscribed in the triangle formed by joining the middle points of the sides, if G be this centre, D the middle point of AB, we have $\angle ADG = 90° - \frac{1}{2}(A-B)$. Now, for the equilibrium of the system, the total resistance at D must obviously be vertical, and act along GD, which it can only do provided λ be not less than $90° - \angle ADG$. The least value of λ is therefore $\frac{1}{2}(A-B)$.



(2). The inclination of AB to the horizon is evidently $\frac{1}{2}(A-B)$, hence the inclination (β) of AC to the horizon is $90° - \frac{1}{2}(A+B) = \frac{1}{2}C$. Hence $\alpha = \beta = \frac{1}{2}C$.

(3). Let S be the strain at C, making an angle θ with the horizon. Taking moments about A, B for the bars AC, BC respectively, we have

$$2S \sin [\tfrac{1}{2}(A+B) - \theta] = W_2 \sin \tfrac{1}{2}C, \quad 2S \sin [\tfrac{1}{2}(A+B) + \theta] = W_1 \sin \tfrac{1}{2}C;$$

whence $$\frac{\sin [\frac{1}{2}(A+B) - \theta]}{\sin [\frac{1}{2}(A+B) + \theta]} = \frac{W_2}{W_1} = \frac{\sin B}{\sin A}; \quad \therefore \theta = \tfrac{1}{2}(A-B).$$

(4). If $$\frac{W_1}{\sin A} = \frac{W_2}{\sin B} = \frac{W_3}{\sin C} = k,$$

then $$2S \sin B = W_2 \sin \tfrac{1}{2}C; \quad \therefore S = \tfrac{1}{2}k \sin \tfrac{1}{2}C.$$

If T, T′ be the strains at A and B,

$$T = \{W_2^2 + S^2 + 2W_2 S \sin \tfrac{1}{2}(A-B)\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2}k \{\sin^2 \tfrac{1}{2}C + 2 \sin B (\sin A + \sin B)\}^{\frac{1}{2}},$$
$$T' = \{W_1^2 + S^2 - 2W_1 S \sin \tfrac{1}{2}(A-B)\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2}k \{\sin^2 \tfrac{1}{2}C + 2 \sin A (\sin A + \sin B)\}^{\frac{1}{2}}.$$

6123. (By Professor BURNSIDE, M.A.)—If the sides *ad*, *bd*, *cd* of a tetrahedron *abcd* meet a surface of the second order in the points a, a_1; b, b_1; c, c_1, respectively; prove that the sections of the surface by the planes bca_1, cab_1, abc_1 are all touched by another plane section of the surface.

Solution by W. J. C. SHARP, M.A.; CHRISTINE LADD; *and others.*

If *bcd*, *cda*, *dab*, *abc* be the planes of reference, the equation to the quadric is

$$Dw^2 + 2Fyz + 2Gzx + 2Hxy + 2Lxw + 2Myw + 2Nzw = 0,$$

and the planes a_1bc, ab_1c, abc_1, abc are

$$Dw + 2Lx = 0, \quad Dw + 2My = 0, \quad Dw + 2Nz = 0, \text{ and } w = 0 \ldots (1),$$

the intersections of which with the plane

$$Dw + 2Lx + 2My + 2Nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

touch the section of the quadric by that plane.

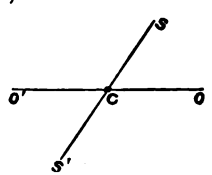
For the equation to the tangent plane at $(x_1\, y_1\, z_1\, w_1)$ reduces to

$$(Gz_1 + Hy_1)\, x + (Fz_1 + Hx_1)\, y + (Fy_1 + Gx_1)\, z = 0,$$

(the tangent plane to the quadric $Fyz + Gzx + Hxy = 0$, which passes through the section,) where the equation (2) and any one of the equations (1) are satisfied by $(xyzw)$ and $(x_1y_1z_1w_1)$.

6538. (By Professor GENESE, M.A.)—Find the envelop of the latera recta of conics having a given director circle, and passing through a given point.

Solution by Professor WOLSTENHOLME, M.A.



Let C be the centre of the director circle, O the given point, then C will be the centre of the ellipse, and if OO′ be bisected in C, OO′ will be a diameter of each of the conics; and if S, S′ be the real foci of one of them, OS . O′S = SO . S′O = excess of square of radius of director circle above CO^2. Hence S, S′ lie on a Cassinian whose foci are O, O′, and the envelop of the latus rectum is the first negative pedal of a Cassinian with respect to its centre, which is a rectangular hyperbola, and the Cassinian is Bernoulli's Lemniscate when the radius of the director circle $= \sqrt{2}$. CO.

5574. (By R. TUCKER, M.A.)—Three particles are projected simultaneously in the same vertical plane, with velocities v_1, v_2, v_3, at inclinations a_1, a_2, a_3 to the horizon; show that, when their directions are parallel,

$$\frac{\sin(a_2 - a_3)}{v_1} + \frac{\sin(a_3 - a_1)}{v_2} + \frac{\sin(a_1 - a_2)}{v_3} = 0.$$

Solution by D. EDWARDES; J. H. TURRELL, M.A.; *and others.*

The vertical velocity of particle (a_1) after time t is $v_1 \sin a_1 - gt$, and its horizontal velocity is $v_1 \cos a$. Let θ be the inclination to horizon of their directions when parallel, then

$$\tan\theta = \frac{v_1 \sin a_1 - gt}{v_1 \cos a_1} = \frac{v_2 \sin a_2 - gt}{v_2 \cos a_2} = \frac{v_3 \sin a_3 - gt}{v_3 \cos a_3};$$

therefore

$$\frac{v_1 \sin a_1 - v_2 \sin a_2}{v_1 \cos a_1 - v_2 \cos a_2} = \frac{v_2 \sin a_2 - v_3 \sin a_3}{v_2 \cos a_2 - v_3 \cos a_3};$$

whence, simplifying, we get the required result.

5918. By Prof. SETH, M.A.)—Prove that the average distance of points within an ellipse from the extremity of the major axis is

$$\frac{2b^2}{3\pi a^2}(3-2e^2) - \frac{2a(1-4e^2)}{3\pi e^3}\sin^{-1}e.$$

Solution by J. HAMMOND, M.A.; ELIZABETH BLACKWOOD; *and others.*

The average distance is $\frac{2}{\pi ab}\iint r\,.\,r\,dr\,d\theta$,

or $\frac{2}{3\pi ab}\int_{-\alpha}^{\alpha} r^3\,d\theta$, where $r\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=\frac{2\cos\theta}{a}$ .....(1)

Put $e=\sin\alpha$, $b=a\cos\alpha$, then (1) gives $r=\frac{2a\cos\theta}{(1+\tan^2\alpha\sin^2\theta}$

Let $\tan\alpha\sin\theta=\tan\phi$, then $\tan\alpha\cos\theta\,d\theta=\sec^2\phi\,d\phi$,
and $r=2a\cos^2\phi$: therefore the average distance is

$$\frac{16a^2}{3\pi b}\int_0^{\alpha}\cos^4\phi\left(1-\frac{\tan^2\phi}{\tan^2\alpha}\right)\cot\alpha\,d\phi$$

$$=\frac{16a^2\cos\alpha}{3\pi\sin^2\alpha}\int_0^{\alpha}\cos^2\phi\sin(\alpha+\phi)\sin(\alpha-\phi)\,d\phi$$

$$=\frac{2a}{3\pi}\int_0^{\alpha}2(1+\cos2\phi)(\cos2\phi-\cos2\alpha)\,d\phi$$

$$=\frac{2a}{3\pi}\int_0^{\alpha}\{\cos4\phi+2\cos2\phi(1-\cos2\alpha)+(1-2\cos$$

$$=\frac{2a}{3\pi}\left\{\frac{\sin4\alpha}{4}+\sin2\alpha(1-\cos2\alpha)+\alpha(1-2\cos^2\right.$$

Now $\sin\alpha=e$, $\cos2\alpha=1-2e^2$, $\sin2\alpha=2e\cos\alpha$,
the average distance is that given in the question.

6553. (By G. F. WALKER, M.A.)—Solve the

$$x^2(y+z)=a^2,\quad y^2(z+x)=b^2,\quad z^2(x+$$

Solution by Rev. J. I. [illegible] M.A.; Bess[illegible]

Multiplying, and a[illegible]

$$x^2y^2z^2(x^2y$$
$$x^2y$$

therefo[illegible]
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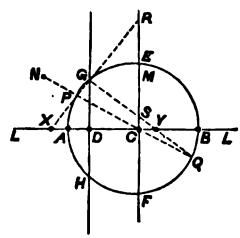
5422. (By Professor TOWNSEND, F.R.S.)—Four points A, B, C, D being supposed given or taken arbitrarily on a straight line L; construct, by elementary geometry, the two X and Y, on the line, the distance of each of which from one of the four D shall be the harmonic mean of its distances from the remaining three A, B, C.

Solution by the PROPOSER.

On the interval AB, between the two extreme points A and B of the given or assumed tetrad A, B, C, D, as diameter, describe a circle, as in the figure, intersecting the two perpendiculars to the given line L at the remaining two points C and D of the tetrad in the two pairs of points E and F, G and H, respectively; and connect either point G of the latter pair with the two required points X and Y, supposed found, on L; the connecting lines GX and GY intersecting the circle at the two second points P and Q, and the perpendicular to L through C at the two points R and S; then since, by



hypothesis, $AX^{-1}+BX^{-1}+CX^{-1} = 3 . DX^{-1}$,

and $AY^{-1}+BY^{-1}+CY^{-1} = 3 . DY^{-1}$,

therefore, by the theory of poles and polars,

$GX^{-1}+PX^{-1}+RX^{-1} = 3 . GX^{-1}$, and $GY^{-1}+QY^{-1}+SY^{-1} = 3 . GY^{-1}$,

therefore $PX^{-1}+RX^{-1} = 2 . GX^{-1}$, and $QY^{-1}+SY^{-1} = 2 . GY^{-1}$,

and therefore GX is the harmonic mean between PX and RX, and GY the harmonic mean between QY and SY; consequently the pencil of four rays which subtends from C the two tetrads of points PRGX and QSGY is harmonic, and therefore GM = GN, where MN is the parallel through G to XY or L. Hence the point N is given, and therefore the two P and Q, and therefore the two X and Y, and therefore, &c.

Denoting by a, b, c the distances of A, B, C, and by x and y those of X and Y from D, it follows at once, from the algebraic solution of the same problem, that the latter are connected with the former by the relations $x+y = 2\,\frac{bc+ca+ab}{a+b+c}$, and $xy = 3\,\frac{abc}{a+b+c}$; which accordingly, regard of course being had to the signs as well as to the magnitudes of the several quantities involved, are the formulæ by which to calculate in numbers the distances of X and Y from D when those of A, B, C from it are given or known.

6591. (By W. H. H. HUDSON, M.A.)—A man standing at the North Pole whirls 24 lbs. Troy weight on a smooth horizontal plane, by a string a yard long, at the rate of 100 turns a minute, and finds that the difference of the forces he has to exert, according as he whirls it one way or the opposite, is roughly 39 grains: find the period of the earth's rotation.

Solution by Rev. T. R. Terry, M.A.; Prof. Matz, M.A.; *and others.*

Let n be the number of turns made per minute by the earth. Then, if we take a foot and a second as the units, we have

$$3\left[\left\{\frac{(100+n)2\pi}{60}\right\}^2-\left\{\frac{(100-n)2\pi}{60}\right\}^2\right]=\frac{39g}{24\times 5760};$$

$$\therefore \text{ approx. } n=\frac{13g}{61440\pi^2}, \quad \therefore \text{ number of turns per day}=\frac{39g}{128\pi^2}.$$

Put $g=32$ and $\pi^2=9{\cdot}9$, and we get, approximately, number of turns per day $=\cdot 996$. Period of earth's rotation $=1$ day 6 min.

6254. (By J. J. Walker, M.A.)—Find at what points on an ellipse the continued product of the semi-diameter, the perpendicular on tangent, and the cosecant of the angle between them, is a minimum.

Solution by R. Knowles, B.A., L.C.P.; G. Eastwood, M.A.; *and others.*

$$\text{The product}=r^3\cdot\frac{d\theta}{dr}=\pm\frac{b^2}{c^2\cos\theta\cdot\sin\theta}=u;$$

$$\frac{du}{d\theta}=\mp\frac{b^2}{c^2}(\operatorname{cosec}^2\theta-\sec^2\theta)=0, \quad \therefore \tan\theta=\pm 1;$$

$$\frac{d^2u}{d\theta^2}=\pm\frac{2b^2}{c^2}\left(\frac{\operatorname{cosec}^2\theta}{\tan\theta}+\tan\theta\cdot\sec^2\theta\right);$$

therefore $\tan\theta=\pm 1$ determines all the points on the curve for which (u) is a minimum.

6080. (By J. F. Moulton, M.A.)—A quantity of fluid fills a paraboloid of latus rectum c to a height h, the axis being vertical and vertex downwards. The density of the fluid varies as the depth. If the fluid pass into a vessel of the form generated round the axis of x, by the curve $a^4y^2=2ch^2x(a-x)(2a-x)$, where a is any constant, the density will vary as (depth)^2.

Solution by the Rev. T. R. Terry, M.A.; Lizzie A. Kittudge; *and others.*

Let large letters refer to old vessel, and small ones to the new. Then

$$\int_0^h \pi Y^2 dX=\tfrac{1}{2}\pi ch^2=\int_0^a \pi y^2 dx;$$

therefore fluid rises to height a in new vessel.

$$\text{Quantity below depth } hD=\int_0^{h(1-D)}\pi Y^2 dX=\tfrac{1}{2}\pi ch^2(1-D)^2,$$

and density at the depth is $\mu k D$;

Quantity below depth ad is $\int_0^{a(1-d)} \pi y^2 dx = \frac{1}{2}\pi c h^2 (1-d^2)^2$;

therefore, equating these expressions, $D = d^2$; therefore the density at depth ad is $\mu h d^2$; that is, it varies as (depth)^2.

6603. (Professor Genese, M.A.)—Given the base and vertical angle of a triangle, find the envelop of the nine-point circle.

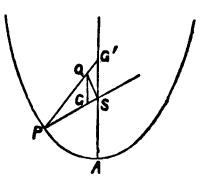
Solution by W. M. Coates, B.A.; E. Rutter; *and others.*

All triangles satisfying the given conditions have the same circumscribing circle, and the diameter of the nine-point circle of a triangle is equal to the radius of the circumscribing circle; moreover, all the nine-point circles pass through the middle point of the given base; consequently the envelop is a circle with the middle point of the base as centre, and radius equal to radius of circumscribing circle.

6512. (By Professor Matz, M.A.)—A heavy prismatic bar of infinitesimal cross-section and length $2c$ rests against the concave arc of a vertical elliptic bowl and a pin placed at the focus; prove that the bar is in equilibrium when its inclination to the vertical is $\cos^{-1}\left\{\frac{1}{e}\left[\frac{b}{a^2c^2} - 1\right]\right\}$, where (a, b) are the axes and e the eccentricity of the ellipse.

Solution by G. F. Walker, M.A.; J. A. Kealy, M.A.; *and others.*

Let the reactions at P and S meet in Q, and let G be the centre of gravity; then QG must be vertical, and therefore parallel to the axis.



Let $\angle PSA = \theta$, and $\angle SPQ = \chi$. Since $SG' = e.SP$, we have $\sin\chi = e\sin(\theta-\chi)$, and $\tan\chi = \frac{e\sin\theta}{1+e\cos\theta}$; and if $SP = r$, since QSP is a right angle and QG is parallel to the axis,

$$(r-c)\sin\theta = (r\tan\chi)\cos\theta, \quad \therefore \left[1 - \frac{c(1+e\cos\theta)}{a(1-e^2)}\right]\sin\theta = \frac{e\sin\theta}{1+e\cos\theta}.\cos\theta,$$

and putting $a(1-e^2) = \frac{b^2}{a}$, $\frac{ac(1+e\cos\theta)}{b^2} = \frac{1}{1+e\cos\theta}$; therefore, &c.

6622. (By T. H. Attwater, M.A.)—If the tangent to an ellipse at P meet the major axis in T, and the tangent at P′ meet the minor axis in T′, and if CP, CP′ are conjugate; prove that TT′ is parallel to an equi-conjugate diameter.

Solution by J. McDowell, M.A.; A. Anderson, B.A.; *and others.*

Let (x', y') be the coordinates of P, then $\frac{ay'}{b}, \frac{bx'}{a}$ will be those of P′; therefore $CT = \frac{a^2}{x'}$, $CT' = \frac{ab}{x'}$; therefore equation of diameter parallel to TT′ is $\frac{x}{a} + \frac{y}{b} = 0$; therefore square of this semi-diameter $= \frac{1}{2}(a^2 + b^2)$.

6588. (By A. Martin, M.A.)—A person is to draw two from the tickets 1, 2, 3 ... 100, all tickets being equally likely. If both tickets be squares, he is to receive £100; if the lower, only £50; if the higher, only £20; find the worth of his chance of gain.

Solution by W. B. Grove, B.A.; Christine Ladd; *and others.*

There are 10 squares among the first hundred numbers; hence the number of ways in which two squares can be drawn $= \frac{10 \cdot 9}{1 \cdot 2} = 45$. The number of ways in which two numbers can be drawn, of which the lower only is a square, is $\sum_{r=1}^{r=10} (90 - r^2 + r) = 570$.

The number of ways, if the higher only is a square, is $\sum_{r=1}^{r=10} (r^2 - r) = 330$; and the total number of ways of drawing two tickets $= 4950$; therefore his expectation $= \frac{1}{4950}(45 \times £100 + 570 \times £50 + 330 \times £20) = £8$.

6525. (By H. McColl, B.A.)—If n quantities be each taken at random between a and $-a$, show that the chance that their sum will be between b and $-b$ is

$$1 - \frac{2}{(2a)^n n!}\left\{(na-b)^n - n(na-b-2a)^n + \frac{n(n-1)}{1 \cdot 2}(na-b-4a)^n - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}(na-b-6a)^n + \ldots\right\} \text{ &c.,}$$

the series to be continued as long as $na - b - 2ra$ is positive.

Solution by G. F. WALKER, M.A.; Rev. U. J. KNISELY, M.A.; *and others.*

The required chance $= \iiint \dots dx_1\, dx_2 \dots dx_n \div (2a)^n$,

where the integral is taken subject to the conditions

$$x_1 + x_2 + x_3 \dots + x_n > -b, \quad < b,$$

and all the quantities between $-a$, $+a$. Add a to each; then, x_1', &c. being $= x_1 + a$, &c., $x_1' + x_2' \dots x_n' > na - b < na + b$, and x_1', &c. lie between 0 and $2a$, *i.e.*, are positive and $< 2a$. Now, the value of the integral

$$\iiint \dots dx_1'\, dx_2' \dots dx_n',$$

with the condition

$x_1' + x_2' + \dots x_n' < na + b$ is $\dfrac{[\Gamma(1)]^n}{\Gamma(n)} \displaystyle\int_0^{na+b} h^{n-1}\, dh$, and this $= \dfrac{(na+b)^n}{n!}$.

If one of the quantities, as x_r', be greater than $2a$, take $x_r' - 2a = y_r$. Then $x_1' + x_2' + \dots y_r + \dots x_n' < na + b - 2a$, and all the quantities are positive, the integral subject to this condition is $\dfrac{(na+b-2a)^n}{n!}$, y_r taking the place of x_r', and so on, if more of the quantities are $> 2a$. Hence the value subject to the restriction that all the quantities are $< 2a$ is

$$\frac{1}{n!}\left[(na+b)^n - n\,(na+b-2a)^n + \frac{n(n-1)}{1.2}(na+b-4a)^n - \dots\right] = \frac{(1)}{n!}.$$

The series, to continue so long as $na + b - 2ra$, is positive. Similarly, the value of the integral, subject to the restriction that $x_1 + x_2 + \dots x_n < na - b$, and all the quantities $< 2a$, is

$$\frac{1}{n!}\left[(na-b)^n - n\,(na-b-2a)^n + \frac{n(n-1)}{1.2}(na+b-4a)^n - \dots\right] = \frac{(2)}{n}.$$

The series, to go on so long as $na - b - 2ra$, is positive. Hence required chance is $[(1) - (2)] \div (2a)^n n!$

Now, considering the coefficient of x^n on the two sides of the expansion of $e^{+bx}(e^{ax} - e^{-ax})^n = e^{bx}(2ax + \dots)^n$, we see that $(1) - (2)$ may be replaced by $(2a)^n n! - 2\,(2)$, and the chance is $1 - \dfrac{2}{(2a)^n n!}\,(2)$.

6556. (By C. LEUDESDORF, M.A.)—If

$3nz^2 = mx^2 - 2lxy$, $3n^2z = l^2y + 2lmx$, $n^3 = l^2m$, prove that $z^3 = x^2y$.

Solution by Prof. SCOTT, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

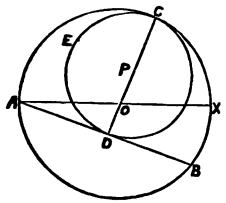
From $3nz^2 = mx^2 + 2lxy$, we get $3\dfrac{z^2}{n^2} = \dfrac{x^2}{l^2} + 2\dfrac{xy}{lm}$;

and from $3nz^2 = l^2y + 2lmx$, we get $3\dfrac{z}{n} = \dfrac{y}{m} + 2\dfrac{x}{l}$;

therefore $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$, and $z^3 = x^2y$.

6585. (By R. E. Riley, B.A.)—Any chord AB is drawn through A, a fixed point of a circle ABC, and a circle CDE is described touching ABC, and the chord AB at its middle point D; prove that the centre of CDE lies on a cardioid, whose pole is the centre of ABC.

Solution by R. Knowles, B.A., L.C.P.; J. O'Regan; *and others.*



Let $OP = r$, $\angle POX = \theta$, $AO = a$;

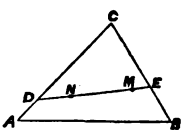
then $2OP = OC - OD$;

therefore $r = \frac{1}{2}a(1-\cos\theta)$,

which is the equation to a cardioid.

6027. (By Professor Seitz, M.A.)—Two points are taken at random in a triangle, and a line drawn through them. If two other points be taken at random in the triangle, show that the chance that they will both fall on the same side of the line, is $\frac{11}{18}$.

Solution by the Proposer.



Let ABC be the triangle, M and N two points such that DE, the line through them, meets AC and BC, and makes an angle with AC less than the angle A.

Let $CD = x$, $DM = y$, $MN = z$, $DE = y'$, $\angle CDE = \theta$, area $CDE = v$, $\dfrac{b^2 \sin C \sin\theta}{2\sin(\theta+C)} = u$, and area $ABC = \frac{1}{2}ab\sin C = 1$.

Then we have $y' = \dfrac{x\sin C}{\sin(\theta+C)}$, $v = \dfrac{x^2 \sin C \sin\theta}{2\sin(\theta+C)}$; an element of the triangle at M is $\sin\theta\, dx\, dy$, and at N it is $d\theta\, z\, dz$.

While θ varies from 0 to A, u varies from 0 to 1; and while x varies from 0 to b, v varies from 0 to u. The limits of y are 0 and y', and those of z are 0 and y, and doubled.

The number of ways two points can be taken in the triangle on the same side of DE is $v^2 + (1-v)^2$; hence, since the whole number of ways the four points can be taken is expressed by unity, we have

$$2\int_0^A \int_0^b \int_0^{y'} \int_0^y [v^2 + (1-v)^2] \sin\theta\, d\theta\, dx\, dy\, z\, dz$$

$$= \tfrac{1}{3}\int_0^A \int_0^b [v^2 + (1-v)^2] \sin\theta\, d\theta\, y'^3\, dx = \tfrac{2}{3}\int_0^1 \int_0^u [v^2 + (1-v)^2]\, u^{-1}\, du\, v\, dv$$

$$= \tfrac{1}{9}\int_0^1 (3 - 4u + 3u^2)\, u\, du = \tfrac{11}{108}.$$

This result is the same as that for each of five other expressions similar to that above. Hence the chance in question is $\frac{11}{18}$.

6394. (By W. J. C. SHARP, M.A.)—Find the area contained between an ordinary catenary, the tractrix which is its involute, and any tangent to the first.

Solution by D. EDWARDES; BELLE EASTON; *and others.*

Measuring s from the vertex, and ϕ from the tangent at the vertex, we have $s = c\tan\phi$; hence the required area is

$$\tfrac{1}{2}\int_0^{\phi} c^2\tan^2\phi\,d\phi = \tfrac{1}{2}c^2(\tan\phi - \phi).$$

[This is an example of the proposition that the area swept over by a cord unwound from the curve, whose intrinsic equation is $s = f(\phi)$, is $\int_{\phi_1}^{\phi_2}\{f(\phi)\}^2\,d\phi$. Similarly $\int_{\phi_1}^{\phi_2}\{f'(\phi)\}^2\,d\phi$ is the area swept over by the radius of curvature.]

5080. (By Professor SYLVESTER, F.R.S.)—If the coordinates (x, y, z) of a curve be proportional to three given cubic functions of t; prove the existence and find the positions of the node, in terms of the coefficients, and extend the method to the case of unicursal curves of any order.

Solution by W. J. C. SHARP, M.A.

If $px = a_0t^3 + a_1t^2 + a_2t + a_3$ and $py = b_0t^3 + b_1t^2 + b_2t + b_3$, the value of pz follows as a consequence of the linear identical relation between x, y, z.

At an ordinary point on the curve, the equations

$$a_0t^3 + a_1t^2 + a_2t + a_3 - px = 0 \text{ and } b_0t^3 + b_1t^2 + b_2t + b_3 - py = 0$$

must have a common root, the condition for which is the non-homogeneous equation to the curve, which is therefore a cubic.

Similarly, at a node, these equations must have two common roots, and the node will be a crunode, a cusp, or an acnode, according as these are real and unequal, equal or imaginary. Hence we have

$$(a_0t + \lambda)(b_0t^3 + b_1t^2 + b_2t + b_3 - py) \equiv (b_0t + \mu)(a_0t^3 + a_1t^2 + a_2t + a_3 - px),$$

therefore $$\begin{Vmatrix} a_0 & , & a_1 & , & a_2 & , & a_3 - px \\ b_0 & , & b_1 & , & b_2 & , & b_3 - py \\ a_0b_1 - a_1b_0, & & a_0b_2 - a_2b_0, & & a_0(b_3 - py) - b_0(a_3 - px), & & 0 \end{Vmatrix} = 0;$$

equations which determine x and y uniquely; and consequently the curve has one and only one node, the coordinates of which are determined by the above equations.

In the general case, it is convenient to proceed somewhat differently.

If $px = (a_0, a_1, a_2 \ldots a_n)(t, 1)^n$, $py = (b_0, b_1, b_2 \ldots b_n)(t, 1)^n$,

then $$pz = (c_0, c_1, c_2 \ldots c_n)(t, 1)^n,$$

where $\lambda a_0 + \mu b_0 + \nu c_0 =$ a constant dependent upon the system of coordi-

nates, and $\lambda a_r + \mu b_r + \nu c_r = 0$. The equation to the tangent at the point t is

$$\begin{vmatrix} (a_1, a_2 \ldots a_n) & (t, 1)^{n-1} & (b_1, b_2 \ldots b_n) & (t, 1)^{n-1} & (c_1, c_2 \ldots c_n) & (t, 1)^{n-1} \\ (a_0, a_1 \ldots a_{n-1}) & (t, 1)^{n-1} & (b_0, b_1 \ldots b_{n-1}) & (t, 1)^{n-1} & (c_0, c_1 \ldots c_{n-1}) & (t, 1)^{n-1} \\ x & & y & & z & \end{vmatrix} = 0,$$

so that the curve is of the $2(n-1)^{\text{th}}$ class at most; and, for a cusp,

$$[a_0, a_1, a_2 \ldots (a_n - px)]\,(t, 1)^n = 0 \text{ and } [b_0, b_1, b_2 \ldots (b_n - py)]\,(t, 1)^n = 0 \ldots (A)$$

must have a square common factor $(t-\alpha)^2$, and $t-\alpha$ will divide the second line of the determinant, so that the class is reduced by one for each cusp, and is $2(n-1)-k$, and the order is n; therefore $2(n-1)-k = n^2-n-2\delta-3k$, and $k+\delta = \frac{1}{2}(n-1)(n-2)$, *i.e.* the deficiency is zero; also if $U = 0$ be the resultant of the equations (A), at a node where they have two common roots $\frac{dU}{dx} = 0$ and $\frac{dU}{dy} = 0$, which equations will determine the nodes, and the common roots satisfy $t^2 \frac{d^2U}{dx^2} - 2\frac{t}{n}\frac{d^2U}{dx\,da_{n-1}} + \frac{1}{n^2}\frac{d^2U}{da_{n-1}^2} = 0$,

so that any node will be a crunode, a cusp, or an acnode, according as

$$\left(\frac{d^2U}{dx\,da_{n-1}}\right)^2 > = < \frac{d^2U}{dx^2}\,\frac{d^2U}{da_{n-1}^2}.$$

Similarly, if $(a_0 . a_1 . a_2 \ldots a_n)(x . 1) = 0$ and $(b_0 . b_1 . b_2 \ldots b_n)(x . 1) = 0$ have a pair of common roots, these will be unequal and real, equal or imaginary, according as $\left(\frac{d^2k}{da_n . da_{n-1}}\right)^2 > = < \frac{d^2k}{da_n^2} \cdot \frac{d^2k}{da_{n-1}^2}$.

[Another solution may be seen in *Reprint*, Vol. XXVI., pp. 83—86.]

6608. (By the Editor.)—Find, to four places of decimals, and, if possible, in a finite form, the value of the following series and its equivalent integral, deduced in Professor Seitz's solution of the Editor's Question 6482 (*Reprint*, Vol. XXXV., p. 30) :—

$$\frac{1}{2^2} - (1-\tfrac{1}{2})\frac{1}{3^2} + (1-\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^2} - (1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})\frac{1}{5^2} + \ldots \equiv \int_0^1 \frac{\log(1-x)}{1+x}\log x\,dx.$$

Solution by W. H. Blythe, B.A.

1. The series $\frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2} + \ldots$

lies between $\frac{1}{n(n-1)(n+1)} + \frac{1}{n(n+1)(n+2)} + \ldots = \frac{1}{2n(n-1)}$

and $\frac{1}{n(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots = \frac{1}{2(n+1)n}$.

If n be less than 25, both these expressions agree to four places of decimals,

$$\frac{1}{25^2} - \frac{1}{26^2} + \ldots = \cdot 0008 \text{ nearly}, \quad \frac{1}{26^2} - \frac{1}{27^2} + \ldots = \cdot 00074 \text{ nearly}.$$

Also, since the general term of the given series is

$$\frac{1}{n}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}+\ldots \text{ to infinity}\right),$$

taking this as lying between $\frac{1}{2n^2(n+1)}$ and $\frac{1}{2n(n+1)(n+2)}$, we get *a fortiori* between $\frac{1}{2(n+1)(n-1)n}$ and $\frac{1}{2n(n+1)(n+2)}$, and hence the sum of all remaining terms after the nth lie between $\frac{1}{4(n-1)n}$ and $\frac{1}{4n(n+1)}$.

We find that, after the sum of 21 terms, these two must agree to four places of decimals, and = ·0006 nearly. Calculating the first 20 terms, using the above approximation for squares after 25^{-2}, and putting ·0006 for the whole remainder of the series, the sum to four places of decimals is found to be ·242894. In fact, the series is as follows:—

·178753	·000841	·000138
·034921	·000610	·000119
·012880	·000438	·000095
·005933	·000341	·000079
·003228	·000255	·000069
·001913	·000210	·000058
·001253	·000160	·000600

I have arranged the series thus:—

$$\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\ldots\right)+\frac{1}{2}\left(\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\ldots\right)+\ldots\frac{1}{n}\left(\frac{1}{(n+1)^2}+\ldots\right).$$

2. *Otherwise*:—The series

$$= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots\right)$$
$$+\frac{1}{2}\cdot\frac{1}{2^2}-\frac{1}{3}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\frac{1}{4^2}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}\right)$$
$$-\frac{1}{5}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}\right)+\frac{1}{6}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}\right)-\ldots$$
$$= \log_e 2\,(1-\tfrac{1}{12}\pi^2)+\ldots = (\cdot6931)(\cdot1775)+\ldots$$
$$= \cdot1231+\cdot125-\cdot0466+\cdot0506-\ldots$$

It will be found that, after a certain number of terms, each is less than the preceding and of opposite sign; the series is convergent, and will be found to agree with the past result, that is, ·2429 nearly.

6285 & 6477. (By Professor Matz, M.A.)—6285. Three points being taken at random in the surface of a circular quadrant; prove that the mean value of all the triangles that can be formed by joining the three points is

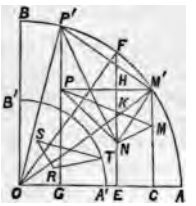
$$\frac{r^2}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

6477. Two points are (1) taken at random on the *arc*, and another point in the *surface* of a circular quadrant; and again, two points are (2) taken at random in the *surface*, and another point on the *arc* of a circular quadrant; prove that the average areas of the triangles formed by joining these points in their respective order will be to each other as

$$45\pi^2 + 84\pi - 684 : 35\pi^2 + 64\pi - 524.$$

Solution by Professor SEITZ, M.A.

(6285.) Let AOB be the quadrant, and M, N, P the random points in its surface. Through M, N, P draw CM′, EF, GP′ perpendicular to AO, EF intersecting MP in K. Now it is necessary to consider only those relative positions of the points in which CM′ lies to the right of GP′, and EF between CM′ and GP′; since under these limitations the triangle will pass through all the possible variations.



Let $OA = r$, $GP = x$, $CM = y$, $EN = z$, $GP' = x'$, $CM' = y'$, $EF = z'$, $EK = w$, $\angle GOP' = \theta$, $\angle COM' = \phi$, $\angle EOF = \psi$, and $v = 1 + (\cos\phi - \cos\theta)$. Then we have $x' = r\sin\theta$, $y' = r\sin\phi$, $z' = r\sin\psi$, $w = xv(\cos\phi - \cos\psi) + yv(\cos\psi - \cos\theta)$; the area of the triangle MNP is $u = \frac{r}{2v}(w - z)$, when $z < w$; and the area is $u_1 = \frac{r}{2v}(z - w)$, when $z > w$.

An element of surface at M is $r\sin\phi\, d\phi\, dy$, at N it is $r\sin\psi\, d\psi\, dz$, and at P it is $r\sin\theta\, d\theta\, dx$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, ϕ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and w, and w and z'.

Hence, since the whole number of ways the three points are taken is $\frac{1}{6}(\frac{1}{4}\pi r^2)^3$, the average area of the triangle is

$$\Delta = \frac{384}{\pi^3 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\left\{\int_0^{w} u\,dz + \int_w^{z'} u_1\,dz\right\} r\sin\theta\, d\theta\, r\sin\phi\, d\phi \cdot r\sin\psi\, d\psi\, dx\, dy$$

$$= \frac{32r^2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\Big\{ 2\sin^3\theta\sin\phi(\cos\phi - \cos\psi)^2 + 2\sin\theta\sin^3\phi(\cos\psi - \cos\theta)^2 + 3\sin^2\theta\sin^2\phi(\cos\phi - \cos\psi)(\cos\psi - \cos\theta) + 3\sin\theta\sin\phi\sin^2\psi(\cos\phi - \cos\theta)^2 - 3\sin^2\theta\sin\phi\sin\psi(\cos\phi - \cos\theta)(\cos\phi - \cos\psi) - 3\sin\theta\sin^2\phi\sin\psi(\cos\phi - \cos\theta)(\cos\psi - \cos\theta)\Big\} \times \sin\theta\sin\phi\sin\psi\, v\, d\theta\, d\phi\, d\psi$$

$$= \frac{16r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\Big\{ 4\sin^2\theta\cos^2\phi + 4\sin^2\phi\cos^2\phi + 4\sin^2\theta\cos^2\theta + 4\sin^2\phi\cos^2\theta - 5\sin^2\theta\cos\theta\cos\phi - 5\sin^2\phi\cos\theta\cos\phi - 6\sin\theta\cos\theta\sin\phi\cos\phi + 12 + 6\cos^2\theta + 6\cos^2\phi - 24\cos\theta\cos\phi - 12\sin\theta\sin\phi - 9(\theta - \phi)\sin(\theta - \phi)\Big\} \times \sin^2\theta\sin^2\phi\, d\theta\, d\phi$$

$$= \frac{8r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}(69\theta + 36\theta\cos\theta - 12\theta\sin^2\theta - 12\theta\sin^4\theta - 60\sin\theta - 45\sin\theta\cos\theta - 10\sin^3\theta\cos\theta + 3\sin^5\theta\cos\theta)\sin^2\theta\, d\theta$$

$$= \frac{r^2}{\pi}\left(\frac{35}{12} + \frac{16}{3\pi} - \frac{131}{3\pi^2}\right).$$

(6477.) (1) Let M′, P′ be two random points in the arc, M′ being nearest A, and N a random point in the surface of the quadrant.

Let $OA = r$, $EN = x$, $EF = x'$, $EH = y$, $\angle AOP' = \theta$, $\angle AOM' = \phi$, $\angle AOF = \psi$, and $v = 1 + (\cos\phi - \cos\theta)$. Then

$$x' = r\sin\psi, \quad y = rv\sin\theta(\cos\phi - \cos\psi) + rv\sin\phi(\cos\psi - \cos\theta);$$

if N is in the segment ACM′, the trapezoid CGP′M′, or the segment GOBP′, the area of the triangle M′NP′ is $u = \frac{r}{2v}(y - x)$; and if N is in the segment M′FP′, the area of the triangle is $u_1 = \frac{r}{2v}(x - y)$.

The limits of θ are 0 and $\frac{1}{4}\pi$; of ϕ, 0 and θ; from $\psi = 0$ to $\psi = \phi$ the limits of x are 0 and x', from $\psi = \phi$ to $\psi = \theta$ they are 0 and y, and y and x', and from $\psi = \theta$ to $\psi = \frac{1}{4}\pi$ they are 0 and x'.

Hence, since the whole number of ways the three points are taken is $\frac{1}{2}(\frac{1}{4}\pi r^2)(\frac{1}{4}\pi r)^2$, the average area of the triangle is

$$\Delta_1 = \frac{32}{\pi^3 r^6}\int_0^{\frac{1}{4}\pi}\int_0^{\theta}\left\{\int_0^{\phi}\int_0^{x'} u\,r\sin\psi\,d\psi\,dx + \int_{\phi}^{\theta}\left(\int_0^{y} u\,dx + \int_y^{x'} u_1\,dx\right) r\sin\psi\,d\psi \right.$$
$$\left. + \int_{\theta}^{\frac{1}{4}\pi}\int_0^{x'} u\,r\sin\psi\,d\psi\,dx\right\} r\,d\theta\,r\,d\phi$$

$$= \frac{8r^2}{3\pi^3}\int_0^{\frac{1}{4}\pi}\int_0^{\theta}\left\{3(\tfrac{1}{4}\pi - 2\theta + 2\phi)\sin(\theta-\phi) - 2\sin\theta + 2\sin\phi + 2\cos\theta \right.$$
$$\left. - 2\cos\phi + 8 - 8\cos(\theta-\phi) + 2\sin^2(\theta-\phi)\right\} d\theta\,d\phi$$

$$= \frac{4r^2}{3\pi^3}\int_0^{\frac{1}{4}\pi}(3\pi - 3\pi\cos\theta + 18\theta + 16\theta\cos\theta - 4\theta\sin\theta + 4 - 32\sin\theta - 4\cos\theta$$
$$- 2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{8r^2}{\pi}\left(\frac{5}{8} + \frac{7}{6\pi} - \frac{19}{2\pi^2}\right).$$

(2) Let N, P be two random points in the surface, and M′ a random point in the arc of the quadrant. Draw the quadrant A′OB′.

Let two points, R, S, be taken at random in the surface of this quadrant, and a third point, T, in its arc.

Let $OA' = x$, and Δ = the average area of the triangle M′NP.

Then for a given value of x the average area of the triangle $RST = x^2\Delta$, $+ r^2$, and the average area of the triangle MNP is

$$\Delta = \int_0^r \frac{x^2}{r^2}\Delta_2 \cdot \tfrac{1}{2}\pi x\,(\tfrac{1}{4}\pi x^2)^2\,dx + \int_0^r \tfrac{1}{2}\pi x\,(\tfrac{1}{4}\pi x^2)^2\,dx = \tfrac{3}{4}\Delta_2.$$

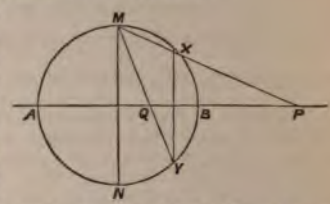
Hence we have $\Delta_2 = \frac{4}{3}\Delta = \frac{r^2}{\pi}\left(\frac{35}{9} + \frac{64}{9\pi} - \frac{524}{9\pi^2}\right)$,

and $\Delta_1 : \Delta_2 = 45\pi^2 + 84\pi - 684 : 35\pi^2 + 64\pi - 524.$

6547. (By E. B. Elliott, M.A.)—From a theorem in plane geometry another is obtained by inversion with regard to any circle in the plane; and from the two theorems, by stereographic projection on a sphere having

for centre and radius the centre and radius of inversion, two theorems of spherical geometry are obtained. Prove that these two are identical.

Solution by Professor TOWNSEND, M.A., F.R.S.



Denoting by M and N the two extremities of the diameter of the sphere perpendicular to the plane of the circle, by P and Q any pair of inverse points on any diameter AB of the circle, and by X and Y the stereographic projections of P and Q from either M or N on the surface of the sphere; then, since, on account of the harmonic section of AB by P and Q, the chord XY is always parallel to the diameter MN of the sphere, all pairs of curves traced out by X and Y on the sphere are reflections of each other with respect to the plane of the circle; therefore, &c.

6162. (By Professor TANNER, M.A.)—Required the form of $\phi(x)$ such that $\frac{x\phi'(y)-y\phi'(x)}{x^2\phi(y)-y^2\phi(x)}$ may be a function of xy.

Solution by J. HAMMOND, M.A.; CHRISTINE LADD; *and others.*

$\frac{x\phi'(y)-y\phi'(x)}{x^2\phi(y)-y^2\phi(x)} = \text{Funct. } (xy)$ is evidently satisfied by $\phi(x) = x^2$.

Assume $\phi(x) = x^2\psi(x)$ for the general solution; then we have

$$\frac{2xy\{\psi(y)-\psi(x)\}+xy\{y\psi'(y)-x\psi'(x)\}}{x^2y^2\{\psi(y)-\psi(x)\}} = \text{Funct. } (xy);$$

or

$$\frac{y\psi'(y)-x\psi'(x)}{\psi(y)-\psi(x)} = \text{Funct. } (xy).$$

Putting $x = 0$, $\frac{y\psi'(y)}{\psi(y)-a} = b$, a and b being arbitrary constants; solving this equation, $\psi(y)-a = cy^b$, and $\phi(x) = x^2\psi(x) = ax^2+cx^{b+2}$.

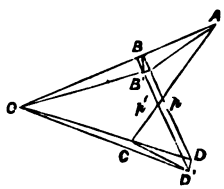
[The PROPOSER remarks that this solution is not, he thinks, complete. The form $\psi x = a+bx^n+cx^{-n}\ldots(1)$ also satisfies the first equation for ψ; a, b, c, n being arbitrary. Moreover, corresponding to the limiting case of $n = 0$, we have a form which, after some reductions, may be written $\psi x = a+b\log x+c(\log x)^2\ldots(2)$, where a, b, c are arbitrary. The property of ϕx is the necessary and sufficient condition that the equation $\frac{dx}{(\phi x)^{\frac{1}{3}}}+\frac{dy}{(\phi y)^{\frac{1}{3}}}\ldots(3)$ should have an integral $x(\phi y)^{\frac{1}{3}}+y(\phi x)^{\frac{1}{3}} = \chi(xy)$. In

fact, the expression given in the question is $= 2\chi' : \chi$. The equation (3) includes as a particular case the well-known equation of the elliptic addition-theorem: viz., putting $n = 2$ in (1), we have $\phi x, = x^2\psi x$, $= c + ax^2 + bx^4$. The method of integration is STURM'S.]

6605. (By Professor PURSER, M.A.)—Four bars are jointed together at the ends, two of them crossing each other; show that, when the four ends lie on the same circle, the difference of the areas of the triangles formed by the bars is less than in any other position.

Solution by G. F. WALKER, M.A.; A. ANDERSON, B.A.; *and others.*

Let AB, BD, DC, CA be the four rods, AC and BD being crossed at p, and let AB, DC meet in O; keep AC fixed, and let BD be displaced to B'D'. This will be about O as instantaneous centre, and BOB' = DOD'. The difference ABp~CnD will be the same as the difference AB'p'~Cp'D' provided the small triangles AB'O, CD'O be equal; that is, if



$$OA \cdot BB' = OC \cdot DD'.$$

But $BB' = OB \cdot \theta$, and $DD' = OD \cdot \theta$, if θ be the small angle turned through; therefore $OA \cdot OB = OC \cdot OD$,

or the four points A, B, C, D lie on a circle.

[The PROPOSER remarks that this no doubt proves that the difference of the areas is stationary when the four points lie on a circle, but that it would appear to require further analysis to show that the *stationary* answered to a minimum position.]

6506. (By Professor CROFTON, F.R.S.)—Express the sum of the series (n being integral) $S = x^n + n^2x^{n-1} + \frac{n^2(n-1)^2}{1 \cdot 2}x^{n-2} + \&c.$

Solution by G. HEPPEL, M.A.; Prof. PRATT, M.A.; *and others.*

To actually *sum* the series is a difficult business, though it is easy enough to *express the sum*, for

$$S = \frac{1}{e^x}\frac{d^n(e^x \cdot x^n)}{dx^n}.$$

NOTE ON QUESTION 6545. *By* ELIZABETH BLACKWOOD.

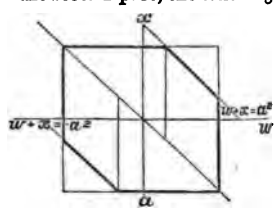
The Solution of this Question, given on p. 61 of this volume, is certainly wrong *on the supposition* $a<2$. The limits of y in the first negative integral imply that $a>(w+x)\,a^{-1}$, which implies that $x<a^2-w$. If the limit a of the variable x supersedes the limit a^2-w (as the integral implies), the former limit must be nearer to x, so that we have $a<a^2-w$, and therefore $w<a^2-a$. If the limit a of the variable w supersedes a^2-a (as the integral also implies), we must have $a<a^2-a$, that is, $a>2$, *which is contrary to the hypothesis.* In the same way the next three integrals may be shewn to be inconsistent with the supposition $a<2$. I may further remark that the solution is *geometrical* as regards the limits of y and z only, being a mere *ipse dixit* as regards the limits of w, x, and the constant a.

NOTE ON QUESTION 6545. *By* G. F. WALKER, M.A.

On referring to my solution, I see that, in the second part, the following correction is required:—

If $a<2$, the hyperbola does not always meet the square, and the point w, x must be confined to the figure bounded by the black line, and *then* this integral *added:*—

$$\int_{a^2-a}^{a}\int_{a^2-w}^{a}\int_{-a}^{a}\int_{-a}^{a} \mathrm{P}\,dx\,dy\,dz.$$



Each of the integrals with respect to x, w splits up into two, making four, taken each over one of the compartments into which the above area is divided.

6639. (By J. J. WALKER, M.A.) — The constants a, b, c of an empirical formula $p=a+b\tan(\theta+c)$ are to be determined from three pairs of observed simultaneous values of the variables, viz., $p\theta$, $p'\theta'$, $p''\theta''$; prove that (1) the single real value of c is that given by

$$\tan c=\Sigma\, p\cos\theta\sin(\theta'-\theta'')\div\Sigma\, p\sin\theta\sin(\theta'-\theta'');$$

and (2), if $\quad (p'-p'')^2\cos(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta')=q\ldots\ldots,$

$$a=(pq+p'q'+p''q'')\div(q+q'+q''),$$

$$b=-(p'-p'')(p''-p)(p-p')\sin(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta')\div(q+q'+q'').$$

Solution by G. HEPPEL, M.A.; BELLE EASTON; *and others.*

1. From the three original equations we obtain

$$\frac{p-p'}{p'-p''}=\frac{(\tan\theta-\tan\theta')(1-\tan\theta''\tan c)}{(\tan\theta'-\tan\theta'')(1-\tan\theta\tan c)};$$

$$\therefore\quad \tan c=\frac{\Sigma\, p(\tan\theta'-\tan\theta'')}{\Sigma\, p\tan\theta(\tan\theta'-\tan\theta'')}=\frac{\Sigma\, p\cos\theta\sin(\theta'-\theta'')}{\Sigma\, p\sin\theta\sin(\theta'-\theta'')}.$$

2. Let $p'-p''=r$, $\sin(\theta'-\theta'')=u$, $\cos(\theta'-\theta'')=v$; then

$$\tan(\theta+c)=\frac{\sin\theta+\cos\theta\tan c}{\cos\theta-\sin\theta\tan c}=\frac{pu+p'u'v''+p''u''v'}{(p''-p')u'u''};$$

and, since $u=-u'v''-u''v'$, we get

$$\tan(\theta+c)=\frac{r''u'v''-r'u''v'}{ru'u''};\quad\text{similarly}\quad\tan(\theta'+c)=\frac{ru''v-r''uv''}{r'u''u};$$

hence

$$\frac{p-a}{p'-a}=\frac{\tan(\theta+c)}{\tan(\theta'+c)}=\frac{rr''uu'v''-r'^2u''uv'}{r^2u'u''v-rr''uu'v'}.$$

Noticing that $r'+r''=-r$, and that $p''r'+p'r''=-pr$, the above equation at once reduces to

$$a[q+q'+q'']=pq+p'q'+p''q'',\quad p-p'=b\left[\frac{r''u'v''-r'u''v'}{ru'u''}-\frac{ru''v-r''uv''}{r'u''u}\right];$$

therefore

$$rr'r''uu'u''=-b(q+q'+q'').$$

6561. (By E. W. Symons, M.A.)—A conic is self-conjugate to the triangle of reference; (x', y', z') are the areal coordinates of its centre, and (x, y, z) the distances of this point from the vertices of the triangle; prove that the radius of the director circle is $(x'x^2+y'y^2+z'z^2)^{\frac{1}{2}}$.

Solution by W. H. Blythe, B.A.; Belle Easton; *and others.*

The equation to the conic is $\frac{\alpha^2}{x_1}+\frac{\beta^2}{y_1}+\frac{\gamma^2}{z_1}=0$; and if this be transformed to rectangular axes through the centre to $Ax^2+2Hxy+By^2=1$, we have

$$A+B\equiv\left(\frac{a^2}{x_1}+\frac{b^2}{y_1}+\frac{c^2}{z_1}\right)\div 4\Delta^2,$$

$$AB-H^2\equiv\frac{1}{4\Delta^2}\left(\frac{a^2b^2\sin^2C}{x_1y_1}+\frac{b^2c^2\sin^2A}{y_1z_1}+\frac{c^2a^2\sin^2B}{z_1x_1}\right)\equiv\frac{1}{x_1y_1z_1\cdot 4\Delta^2}.$$

The square on the radius of the director circle equals sum of squares on semi-axes $=\frac{A+B}{AB-H^2}=a^2y_1z_1+b^2x_1z_1+c^2x_1y_1$. If α, β, γ; $\alpha_1, \beta_1, \gamma_1$, be two points, the distance between them is the square root of

$$\frac{b^2+c^2-a^2}{2}(\alpha-\alpha_1)^2+\frac{c^2+a^2-b^2}{2}(\beta-\beta_1)^2+\frac{a^2+b^2-c^2}{2}(\gamma-\gamma_1)^2.$$

Hence, if x be the distance between (x_1, y_1, z_1) and $(1, 0, 0)$,

$$x^2=\frac{b^2+c^2-a^2}{2}(x_1-1)^2+\frac{c^2+a^2-b^2}{2}y_1^2+\frac{a^2+b^2-c^2}{2}z_1^2,$$

with similar expressions for y^2 and z^2. Multiply x^2 by x_1, y^2 by y_1, z^2 by

z_1, and, remembering that $x_1+y_1+z_1=1$, we get

$$x_1x^2+y_1y^2+z_1z^2=\frac{b^2+c^2-a^2}{2}(y_1x_1+z_1x_1)+\text{two similar terms}$$
$$=a^2y_1z_1+b^2x_1z_1+c^2x_1y_1,$$

which has been shown $=r^2$, where $r=$ radius of director circle;

hence we have $r=(x_1x^2+y_1y^2+z_1z^2)^2$.

6638. (By H. G. Dawson, M.A.)—If four of the roots of

$$(a,\ b,\ c,\ d,\ e,\ f)\,(x,\ 1)^5=0$$

be connected by the relation $\alpha+\beta=\gamma+\delta$, prove that the remaining root ϵ is given by the equations $z=a\epsilon+b$, $z^3-8\mathrm{H}z+16\mathrm{G}=0$, where $\mathrm{H}=b^2-ac$, $\mathrm{G}=2b^3-3abc+a^2d$.

Solution by the Rev. T. R. Terry, F.R.A.S.; J. O'Regan; *and others.*

Multiplying throughout by a^4, and putting $y=ax+b$, we get

$$y^5-10\mathrm{H}y^3+10\mathrm{G}y^2+\ldots=0.$$

One root of this equation is z, and the others are connected by the relation

$$\alpha+\beta=\gamma+\delta,\quad\text{therefore}\quad\alpha+\beta=\gamma+\delta=-\tfrac{1}{2}z,$$
$$-10\mathrm{H}=\alpha\beta+\gamma\delta-\tfrac{3}{4}z^2,\quad-10\mathrm{G}=\tfrac{1}{2}z(\alpha\beta+\gamma\delta)+\tfrac{1}{4}z^3.$$

Hence $$\mathrm{H}z+16\mathrm{G}=0.$$

6314. (By Professor Seitz, M.A.)—In the surface of a circle two lines are drawn at random in length and direction; prove that the chance that they intersect is $\frac{3}{16}$.

Solution by the Proposer; Prof. Matz, M.A.; *and others.*

Let AB and CD be two chords intersecting in P, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD.

Let $\mathrm{OA}=r$, $\mathrm{PA}=x$, $\mathrm{PB}=y$, $\angle\mathrm{AOH}=\theta$, $\angle\mathrm{HOK}=\phi$, and $\angle\mathrm{COK}=\psi$. Then $\mathrm{AB}=2r\sin\theta$, $\mathrm{CD}=2r\sin\psi$, and

$$xy=r^2[\sin^2\theta-(\cos\psi\,\mathrm{cosec}\,\phi-\cos\theta\cot\phi)^2].$$

The number of ways two lines can be drawn, one in AB and the other in CD, so as to intersect, is $4x^2y^2$, and the whole number of ways they can be drawn in these chords is $\mathrm{AB}^2\,.\,\mathrm{CD}^2=16r^4\sin^2\theta\sin^2\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; and for intersection the limits of ψ must be $\pm(\theta-\phi)$ and $\theta+\phi$ (the double sign being taken positive when $\phi<\theta$, and negative when $\phi>\theta$).

Hence the required chance is

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} 4x^2y^2\, r\sin\theta\, d\theta\, d\phi\, r\sin\psi\, d\psi + \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi} 16r^6\sin^3\theta\sin^3\psi\, d\theta\, d\psi$$

$$= \frac{9}{32\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} [\sin^2\theta - (\cos\psi\,\operatorname{cosec}\phi - \cos\theta\cot\phi)^2]^2 \sin\theta\, d\theta\, d\phi\sin\psi\, d\psi$$

$$= \frac{3}{10\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \sin^6\theta\sin\phi\, d\theta\, d\phi = \frac{3}{10\pi}\int_0^{\frac{1}{2}\pi}\sin^6\theta\, d\theta = \frac{3}{32}.$$

6656. (By W. H. Besant, F.R.S.) — Find (1) the volume; (2) the surface generated by the revolution about the initial line of the curve $r = a(1+\cos\theta)$; also (3) the mean value of the radii to all points within the volume enclosed; and prove (4) that the mean value of the radii to the surface is equal to a when the radii are drawn symmetrically in space, and to $\frac{16}{15}a$ when the surface is covered equably by the ends of the radii; also (5) that, if the volume be made up of a series of shells of equal volumes, bounded by a succession of surfaces similar and similarly situated to the outer surface, and having the same cuspidal point, the mean value of their surfaces is three-fifths of the outer surface.

Solution by the Rev. T. R. Terry, F.R.A.S.; Prof. Droz; *and others.*

Let M_1, M_2, M_3, M_4 be the four mean values to be found; then

(1) $\quad V = \text{volume} = \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r\sin\theta\,.\,r\,d\theta\,dr = \frac{8\pi a^3}{3}.$

(2) Since element of arc $= 2a\cos\frac{1}{2}\theta\, d\theta$, therefore

$$S = \text{surface} = \int_0^{\pi} 2\pi r\sin\theta\,.\,2a\cos\frac{\theta}{2}\,d\theta = \frac{32\pi a^2}{5}.$$

(3) $\quad M_1V = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,.\,2\pi r\sin\theta\,.\,r\,d\theta\,dr = \frac{16\pi a^4}{5}, \quad \therefore\ M_1 = \frac{6a}{5}.$

(4) $\quad 4\pi M_2 = \int_0^{\pi} r\,.\,2\pi\sin\theta\, d\theta = 4\pi a, \quad$ therefore $M_2 = a.$

$$M_3S = \int_0^{\pi} r\,.\,2\pi r\sin\theta\,.\,2a\cos\frac{\theta}{2}\,d\theta = \frac{64\pi a^3}{7}, \quad \text{therefore } M_3 = \frac{10a}{7}.$$

(5) Since $S = \lambda V^{\frac{2}{3}}$, where λ is a constant, we have

$$M_4V = \int_0^{V} \lambda V^{\frac{2}{3}}\,.\,dV = \tfrac{3}{5}\lambda V^{\frac{5}{3}}, \text{ therefore } M_4 = \tfrac{3}{5}S.$$

6646. (By W. B. Grove, B.A.)—There are 24 cards, every one of which is painted with one or more of the three colours—red, blue, and yellow. Observation informs us that there are 15 painted with blue, 19

with yellow, and 16 not painted with red; 8 painted with both red and blue, 3 with both red and yellow, 10 with both blue and yellow. Find the law or laws which regulate the combinations of the colours.

Solution by ALEXANDER MACFARLANE, D.Sc., F.R.S.E.

Let U denote the 24 cards, r the portion which are painted red, b the portion which are painted blue, y the portion which are painted yellow. Then the data are

$$U = 24, \quad b = \frac{15}{24}, \quad y = \frac{19}{24}, \quad 1-r = \frac{16}{24}, \quad rb = \frac{8}{24} \ldots (1, 2, 3, 4, 5),$$

$$ry = \frac{3}{24}, \quad by = \frac{10}{24}, \quad (1-r)(1-b)(1-y) = 0 \ldots\ldots (6, 7, 8).$$

From (5) and (4), we have $rb = \frac{8}{24} = r$.....(9).

From (3), (6), (4), $y(1-r) = y - yr = \frac{19}{24} - \frac{3}{24} = \frac{16}{24} = 1-r$.....(10).

From (2), (7), (3), $b(1-y) = b - by = \frac{15}{24} - \frac{10}{24} = \frac{5}{24} = 1-y$.....(11),

But (11) follows from (9) and (10);

for $r = \frac{0}{1-b}$ from (9), and $r = \frac{1-y}{1-y}$ from (10),

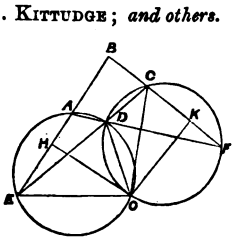
therefore $\frac{0}{1-b} = \frac{1-y}{1-y}$, therefore $0 = (1-b)(1-y)$,

which is equivalent to (11).

Hence the laws are as follows:—A card which is red is also blue; and a card which is not red, is yellow.

6641. (By W. S. MCCAY, M.A.)—Prove that the quadrilinear coordinates of the focus of the parabola touching four lines x_1, x_2, x_3, x_4 are given by the equations $R_1 x_1 = R_2 x_2 = R_3 x_3 = R_4 x_4$, where R_1 is the radius of the circle circumscribing the triangle $x_2 x_3 x_4$, &c.

I. *Solution by* W. M. COATES, B.A.; LIZZIE A. KITTUDGE; *and others.*



Let ABCD be the quadrilateral formed by the four lines x_1, x_2, x_3, x_4; and O the focus of the parabola touching them.

Let fall perpendiculars OH and OK from O on AB and BC. Join OE, OC, and OD.

Now, since the circles circumscribing the four triangles formed by x_1, x_2, x_3, x_4 pass through O, the triangles OHE and OCK are equiangular, therefore

$$OH : OK = OE : OC = R_2 : R_1;$$

therefore $x_1 : x_2 = R_2 : R_1$, or $x_1 R_1 = x_2 R_2$.

Similarly for the others; therefore x_1, x_2, x_3, x_4 are given by

$$x_1 R_1 = x_2 R_2 = x_3 R_3 = x_4 R_4.$$

II. *Solution by the* PROPOSER.

The theorem will be proved if I show that the parameter of a parabola, which is inscribed in a triangle, and the coordinates of whose focus are (α, β, γ), is given by the equation $\frac{1}{m^2} = \frac{2R}{\alpha\beta\gamma}$, as most readily appears thus. For any point we have

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \Delta R^{-1},$$

where Δ is the area of the triangle, and R the radius of the circumscribed circle.

Reciprocate the triangle to a circle of unit radius whose centre is $\alpha\beta\gamma$; the distances of the origin to the new vertices will be connected with the old triangle by the relation

$$\frac{\sin A}{r_1} + \frac{\sin B}{r_2} + \frac{\sin C}{r_3} = \frac{\Delta}{R},$$

and A, B, C being equal (because $\alpha r_1 = \beta r_2 = \gamma r_3$), or supplemental to the angles between r_1, r_2, r_0, this becomes

$$2\Delta' = \frac{\Delta}{R} \cdot \frac{1}{\alpha\beta\gamma},$$

where Δ' is the area of the new triangle. (This also follows from TOWNSEND'S *Modern Geometry*, Vol. I., p. 234.)

If the origin be on the circumscribed circle of the old triangle, the two triangles are similar, and

$$\frac{\Delta'}{\Delta} = \frac{R'^2}{R^2}, \quad \text{hence} \quad R'^2 = \frac{R}{2\alpha\beta\gamma},$$

and the circumscribed circle of the new triangle being the reciprocal of the parabola in question, $2R'm = 1$, giving the required value of m.

The relations in the question may also be obtained by elementary geometry; they were first arrived at by elimination.

6615. (By E. B. ELLIOTT, M.A.)—Prove that

$$\int_0^a x^n (2a-x)^n \, dx = 2^{2n} \int_0^a x^n (a-x)^n \, dx \quad \ldots\ldots (1),$$

$$\frac{2^{2n}-1}{\Gamma(2n+2)} = \frac{1}{\Gamma(2n+1)} + \frac{1}{1.2} \cdot \frac{1}{\Gamma(2n)} + \frac{1}{1.2.3} \cdot \frac{1}{\Gamma(2n-1)} \quad \ldots\ldots (2);$$

the series stopping at the n^{th} term if n be a positive integer or zero.

Solution by R. KNOWLES, B.A., L.C.P.; D. EDWARDES; *and others.*

1. The right-hand expression

$$\equiv 2^{2n} \cdot a^{2n+1} \cdot \frac{\Gamma(n+1)\,\Gamma(n+1)}{\Gamma(2n+2)} = 2^{2n} \cdot a^{2n+1} \frac{n!\,n!}{(2n+1)!}.$$

Putting $x = a(1-\sin\theta)$, the left-hand expression

$$\equiv a^{2n+1}\int_0^{\pi}\cos^{2n+1}\theta\,d\theta = \frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3}a^{2n+1} = 2^{2n}a^{2n+1}\cdot\frac{n!\,n!}{(2n+1)!}.$$

2. Since the expansion of $(1+1)^{2n+1}$ contains $2n+2$ terms, we have

$$2^{2n}-1 = \tfrac{1}{2}(1+1)^{2n+1}-1 = (2n+1)+\frac{(2n+1)(2n)}{1.2}+\ldots \text{ to } n \text{ terms};$$

$$\therefore\quad \frac{2^{2n}-1}{\Gamma(2n+2)} = \frac{1}{\Gamma(2n+1)}+\frac{1}{1.2}\,\frac{1}{\Gamma(2n)}+\frac{1}{1.2.3}\,\frac{1}{\Gamma(2n-1)}+\ldots\ldots$$

6202. (By the Rev. W. A. Whitworth, M.A.)—Give, in as simple a form as possible, the arithmetical rule for finding the seventh root of a given number.

I. *Solution by the* Proposer.

1. Mark off the figures into periods of 7 digits, making the unit figure the last one of a period.

2. Find the greatest seventh power in the first period, and subtract: bring down the second period, and put the root just found into the quotient.

3. Proceed to form a trial divisor as follows:—

A. Put down 7 times the sixth power of the quotient, and add 6 cyphers to it. Judge by division what the new figure of the quotient must be.

B. Square the quotient as now completed, and multiply it by 14 times the new figure and by the cube of the quotient minus the new figure.

C. Cube the quotient as now completed, and multiply it by 7 times the square of the new figure and by the quotient minus the new figure.

D. Add the fifth power of the new figure to seven times the fifth power of the quotient minus the new figure, and multiply by the new figure.

4. Add together the results of A, B, C, D, treat the sum as a divisor, multiply by the new figure, subtract and take down a new period, and proceed as before, paragraph 3, continuing the process until all or a sufficient number of figures are found.

[The foregoing solution gives what Mr. Whitworth states that he required; but the following solution has been sent by many correspondents.]

II. *Solution by* J. O'Regan; E. Rutter; *and others.*

Let N denote any number whose root is to be found, n the index of the root, a its nearest rational root, or a^n the nearest rational power to N, whether greater or less, x the remaining part of the root sought, which may be either positive or negative; then the required root is given by

$$N^{\frac{1}{n}} = \frac{(n+1)N+(n-1)a^n}{(n-1)N+(n-1)a^n}a;$$

that is, in words, to $n+1$ times N add $n-1$ times a^n, and to $n-1$ times N add $n+1$ times a^n, then the former sum multiplied by a and divided by the latter sum will give the root nearly.

By substituting for n in the general formula severally the numbers 2, 3, 4, 5, &c., we shall get the following particular expressions, as adapted to the 2nd, 3rd, 4th, 5th roots, &c. :—

$$\frac{3N+a^2}{N+3a^2}.a = N^{\frac{1}{2}}, \quad \frac{2N+a^3}{N+2a^3}.a = N^{\frac{1}{3}}, \quad \frac{5N+3a^4}{3N+5a^4}.a = N^{\frac{1}{4}},$$

$$\frac{3N+2a^5}{2N+3a^5}.a = N^{\frac{1}{5}}, \quad \frac{7N+5a^6}{5N+7a^6}.a = N^{\frac{1}{6}}, \quad \frac{4N+3a^7}{3N+4a^7}.a = N^{\frac{1}{7}}.$$

Example 1.—To find the square root of 365; $N = 365$, $n = 2$, the nearest square is 361, whose root is 19; hence we have

$$\frac{3N+a^2}{N+3a^2}.a = \frac{1456}{1448}.19 = 19{\cdot}10497, \text{ \&c.}$$

To approach still nearer, substitute this last found root for a, the values of the other letters remaining as before; then we have

$$a^2 = \frac{19^2 \times 182^2}{181^2} = \frac{3458^2}{181^2}; \text{ hence } \frac{3N^2+a^2}{N^2+3a^2}.a = \frac{3458 \times 47831059}{181 \times 47831057},$$

which, being brought into decimals, gives the square root of 365 true to twenty places of decimals.

Example 2.—To find the seventh root of 126·2; we have $N = 126{\cdot}2$, $n = 7$, the nearest root $a = 2$, also $a^7 = 128$; hence $4N+3a^7 = 888{\cdot}8$, $3N+4a^7 = 890{\cdot}8$; therefore

$\frac{8888}{8908} = 1{\cdot}995957$ = the root very exact by one operation.

Example 3.—To find the 365th root of 1·05, or the amount of *one pound* for 1 day at 5 per cent. per annum compound interest. Here $N = 1{\cdot}05$, $n = 365$, $a = 1$ the nearest root; hence $366N+364a = 748{\cdot}3$, and $364N+366a = 748{\cdot}2$; hence $748{\cdot}2 : 748{\cdot}3 = 1 : \frac{7483}{7482} = 1{\cdot}00013366$, the root required, very exact, by *one operation*.

6208. (By E. W. Symons, M.A.)—Find the equation of the lines that join the origin to the points of intersection of the curves

$$u_1+u_2+\ldots+u_m = 0, \quad v_1+v_2+\ldots+v_n = 0.$$

Solution by W. J. C. Sharp, M.A.; W. Griffith; *and others.*

Form the eliminant of

$$z^{m-1}u_1+z^{m-2}u_2+\ldots+u_m = 0, \quad z^{n-1}v_1+z^{n-2}v_2+\ldots+v_n = 0;$$

and the result, which will be of constant weight mn, will be the equation

required. In the same way, eliminant of

$$x^m u_0 + x^{m-1} u_1 + \ldots + u_m = 0, \quad x^n v_0 + x^{n-1} v_1 + \ldots + v_n = 0$$

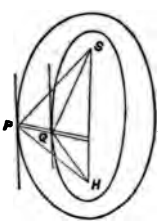
will be the equation to the lines, when the origin is not upon the curves.

If u_1, u_2, &c., v_1, v_2, &c., be ternary quantics in xyz, the same eliminant will give the equation to the cone standing upon the intersection of the surfaces.

If the same line pass through two intersections, this will be a double line on cone, and so on.

6620. (By G. F. Walker, M.A.)—Prove that the line of quickest descent, from one to the other of two confocal ellipses whose major axis is vertical, subtends equal angles at the foci.

Solution by A. Anderson, B.A., *and others.*



Let PQ be the line; $\angle PSH = S_1$, $\angle QSH = S_2$; then, since the normals are parallel, we have

$$S_1 + \tfrac{1}{2}P = S_2 + \tfrac{1}{2}Q, \text{ and } H_1 + \tfrac{1}{2}P = H_2 + \tfrac{1}{2}Q;$$

therefore $S_1 - H_1 = S_2 - H_2$,

and $S_1 - S_2 = H_1 - H_2$,

which proves the theorem. A concentric rectangular hyperbola can be drawn through PQSH.

4706. (By S. Tebay, B.A.)—A bag contains n tickets, marked with the numbers 1, 2, 3, ... n, representing prizes of the respective values a_1, a_2, a_3, ... a_n. A person draws m tickets, and selects the highest, the rest being replaced; a second person draws m tickets, and selects the highest, the rest being replaced; and so on for three or more persons. Find the expectations.

Solution by the Proposer.

Let $N_r = \dfrac{(n-r)!}{m!\,(n-m-r)!}$, $M_r = \dfrac{(m+r-1)!}{(m-1)!\,r!}$. The number m can be selected in one way, and the expectation is $\dfrac{1}{N} a_m$; $m+1$ can be selected in $m = M_1$ ways, and the expectation is $\dfrac{M_1}{N} a_{m+1}$; $m+2$ can be selected in $\dfrac{m(m+1)}{1.2} = M_2$ ways, and the expectation is $\dfrac{M_2}{N} a_{m+2}$; and so on. Hence the whole expectation is

$$\frac{1}{N}(a_m + M_1 a_{m+1} + M_2 a_{m+2} + \ldots + M_{n-m} a_n).$$

The chance that m was selected is $\frac{1}{N}$, and the expectation, as before, is

$$\frac{1}{NN_1}(a_{m+1} + M_1 a_{m+2} + \ldots + M_{n-m-1} a_n).$$

The chance that $m+1$ was selected is $\frac{M_1}{N}$, and the expectation is

$$\frac{1}{NN_1}(a_m + M_1 a_{m+2} + \ldots + M_{n-m-1} a_n);$$

and similarly for $m+2$, $m+3$, &c.; therefore the whole expectation is

$$\frac{1}{NN_1}[M_1 + M_2 + \ldots + M_{n-m-1}]\, a_m$$

$$+\frac{1}{NN_1}[1 + M_1(M_2 + M_3 + \ldots + M_{n-m-1})]\, a_{m+1}$$

$$+\frac{1}{NN_1}[M_1(1 + M_1) + M_2(M_3 + M_4 + \ldots + M_{n-m-1})]\, a_{m+2}$$

$$+\frac{1}{NN_1}[M_2(1 + M_1 + M_2) + M_3(M_4 + M_5 + \ldots + M_{n-m-1})]\, a_{m+3} + \&c.$$

In like manner, by supposing every two of the numbers m, $m+1$, &c., to have been selected, we can find the expectation for the third drawing; and so on for the fourth, &c.

If a_m, a_{m+1}, &c., be equal, the expectation is a_m, and we have

$$\frac{n!}{m!\,(n-m)!} = 1 + m + \frac{m(m+1)}{1.2} + \ldots + \frac{(n-1)!}{(m-1)!\,(n-m)!}.$$

4994. (By Professor SYLVESTER, F.R.S.)—The coordinates of a unicursal quartic x, y, z are given proportional to PQ′, P′Q, QQ′, where

$P = a + bt + ct^2$, $P' = a' + b't + c't^2$, $Q = \alpha + \beta t + \gamma t^2$, $Q' = \alpha' + \beta' t + \gamma' t^2$.

Show that two of the nodes are the intersections of xy with z, and that the coordinates of the third node are in the proportion of LM′ : L′M : MM′, where L, M, L′, M′ are the four first minors taken in a certain order of the rectangular matrix

$$\begin{vmatrix} a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}$$

Solution by W. J. CURRAN SHARP, M.A.

$$\mu x = PQ' = (a + bt + ct^2)(a' + \beta' t + \gamma' t^2),$$
$$\mu y = P'Q = (a' + b't + c't^2)(a + \beta t + \gamma t^2),$$
$$\mu z = QQ' = (a + \beta t + \gamma t^2)(a' + \beta' t + \gamma' t^2).$$

Hence $x = 0$, $z = 0$ gives $Q' = 0$; or t has two distinct values at this point, which is therefore a node; similarly $y = 0$, $z = 0$ is a node. At the third node the same values of t must come from

$$x(a + \beta t + \gamma t^2) = z(a + bt + ct^2), \quad y(a' + \beta' t + r' t^2) = z(a' + b't + c't^2);$$

therefore $$\frac{\gamma x - cz}{\gamma' x - c' z} = \frac{\beta x - bz}{\beta' y - b' z} = \frac{ax - az}{a' y - a' z};$$

$$\therefore \quad x \begin{vmatrix} a' . a . a' \\ b' . \beta . \beta' \\ c' . \gamma . \gamma' \end{vmatrix} = z \begin{vmatrix} a' . a' . a \\ \beta' . b' . b \\ \gamma' . c' . c \end{vmatrix}, \quad y \begin{vmatrix} a . a' . a \\ \beta . \beta' . b \\ \gamma . \gamma' . c \end{vmatrix} = z \begin{vmatrix} a . a' . a \\ \beta . b' . b \\ \gamma . c' . c \end{vmatrix}.$$

or $$x\mathrm{M} = z\mathrm{L} \text{ and } y\mathrm{M}' = z\mathrm{L}';$$

and therefore $$x : y : z = \mathrm{LM}' : \mathrm{L}'\mathrm{M} : \mathrm{MM}'.$$

6472. (By Professor Townsend, F.R.S.)—Two normals to an ellipse being supposed to intersect at right angles on the curve; required the limits to their ratio requisite to their being both maxima vectors from their point of intersection to the curve.

Solution by the Proposer.

Denoting by r and s the lengths of the two normal chords, by p and q those of the two perpendiculars from the centre upon the tangents at their normal extremities, by ρ and σ the radii of the two circles of curvature at the same points, and by a and b the two semi-axes of the curve; then, the two normal chords determining manifestly, with the connector of their normal extremities, a triangle of maximum area inscribed to the ellipse, we have consequently $p = \frac{3}{4}r$, $q = \frac{1}{4}s$, and $pq = \frac{1}{4}(3\sqrt{3})\,ab$, and therefore $\rho = \frac{a^2b^2}{p^3} = \frac{1}{2}\frac{s^2}{r}$, and $\sigma = \frac{a^2b^2}{q^3} = \frac{1}{2}\frac{r^2}{s}$; hence, in order that ρ should be $< r$ and that σ should be $< s$, which are requisite in order that r and s should be both maxima vectors from their common extremity to the curve, we must have $s+r$ and $r+s$ both $< \sqrt{2}$; therefore, &c.

6677. (By Dr. Macfarlane, F.R.S.E.)—(1) A lady, on being asked about a photograph in her album, gave the following answer:—"You know that I have no daughters; that person's daughter's son was the father of a grandchild of mine." [Also solve, by a like method, the two following problems:—

(2) "If Dick's father is Tom's son, what relation is Dick to Tom?"
(3) "Sisters and brothers I have none, but that man's father is my father's son."]

Solution by the Proposer.

1. I have proposed this question for the purpose of illustrating an extension of the Algebra of Logic which I have recently made in several papers contributed to the Royal Society of Edinburgh on an *Algebra of Relationship*.

Let A denote the lady, X the person referred to. Let c denote child, a subscript m denote male, and a subscript f denote female. Then c^{-1} is the proper expression for parent. Σ is used to denote *all*.

The first condition gives

$$\Sigma_f c_f A = 0, \text{ and the second } {}_m c_f X = {}_m c^{-1} c c_f A \ldots\ldots\ldots\ldots (1, 2).$$

From (1) and (2), we have

$$\Sigma_m c_f A = \Sigma c_f A, \quad X = c^{-1} {}_f c^{-1} {}_m c^{-1} c c_f A = c^{-1} {}_f c^{-1} {}_m c^{-1} c_m c_f A;$$

but ${}_m c^{-1} c_m = {}_m$ always, therefore $X = c^{-1} {}_f c^{-1} {}_m c_f A$; but ${}_f c^{-1} c_f = {}_f$ always, therefore $X = c^{-1} {}_f A$; that is, X was either the father or the mother of the lady.

The problems mentioned in the editorial note are solved as follows:—

2. Let D denote Dick and T denote Tom, then the given relation is expressed by the equation ${}_m c^{-1} {}_m D = {}_m c_m T$, that is, the male parent of the man D is identical with a male child of the man T. It is required to transform this equation so that D may be left by itself. The rule given in my papers on an *Algebra of Relationship* for such a transformation is as follows:—A symbol may be removed from the front of one side of an equation, provided its reciprocal be placed in front of the other side, the reciprocals of the sex-symbols m and f being m and f respectively.

Hence $D = {}_m c_m c\ T$; that is, Dick is a son of a son of the man Tom. We may drop one or more of the sex-symbols if we choose; thus $D = {}_m c^2 T$, that is, Dick is a grandson of Tom; or $D = c^2 T$, that is, Dick is a grandchild of Tom.

Suppose that we have given, in addition, that Bill's daughter is Tom's mother. What is the relation of Dick to Bill?

The equations are ${}_m c^{-1} {}_m D = {}_m c_m T$, ${}_f c_m B = {}_f c^{-1} {}_m T$.. .....(1, 2). From (1) and (2), $D = {}_m c_m c_m T$, $T = {}_m c_f c_m B$, therefore $D = {}_m c_m c_m c_f c_m B$; that is, Dick is a son of a son of a son of a daughter of Bill. This involves that $D = {}_m c^4 B$, that is, Dick is a great-great-grandson of B; or $D = c^4 B$, that is, Dick is a great-great-grandchild of Bill.

3. Let A denote the speaker, and X the person referred to. Then the first clause gives us the equation $\Sigma c c^{-1} A = 2A$ .....(1), that is, all the children of the parents of A is A; and the second clause gives the equation ${}_m c^{-1} {}_m X = {}_m c_m c^{-1} A$.....(2). From (1) it follows that $\Sigma c_m c^{-1} A = A$, therefore from (2) ${}_m c^{-1} {}_m X = {}_m A$, that is, A was the father of the man X; or $X = {}_m c_m A$, that is, X was a son of the man A.

6634. (By Elizabeth Blackwood.)—A straight line is divided into n random segments, show that the chance that these segments can form an n-sided polygon is $1 - n 2^{1-n}$.

Solution by the Proposer.

Postulate:—To divide any magnitude into n random parts, take n random points in the circumference of any circle, and divide the given magnitude into n parts proportional to the n random arcs into which the circumference is thus divided.

Let x denote the statement, *Some one of the n random arcs into which the circumference is divided will be greater than the sum of the n − 1 others*, that is, *greater than half the circumference*. We are required to find $1 - x_n$, in which x^c denotes *the chance that x is true*.

Measuring throughout in the *positive* direction of the circumference, let a denote the statement, (1) *Every one of the* $n-1$ *points* B, C, D, &c. *will be less than half the circumference distant from* A; let b denote the statement, (2) *Every one of the* $n-1$ *points* A, C, D, &c. *will be less than half the circumference distant from* B; and so on for every point in succession. Then, evidently, $x = a+b+c+\ldots$ Hence

$$x_s = (a+b+c+\ldots)_s = a_s+b_s+c_s+\ldots = na_s = n\,2^{1-n},$$

for only one of the n statements a, b, c, &c. can be true, and they are all equally probable. Thus, the required chance is $1-n\,2^{1-n}$.

6610. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—

To a wee planetoid, but recently out,
I am bound to dispatch an opinion,
On how to effect a design they're about
Of improving their little dominion.
Tired of their islands, they long for a Continent:
Here is the statement they give me, their want anent:—

Their cities line their woody shore,
And creeks, that curl around in curious ways,
Beating the bays
Of Hellas or of Scotland;
So that of land is little more
About them than of not-land.

To no one city can there be
Of home approaches more than sacred three.
Each has two shore lines, and the only other
Is ferry o'er the creek to fronting town,
Or high-road, bound of province, to another,
Washed by another inlet. Peer and clown,
The laws from olden time forbidding byroad,
Must travel by that shore, or boat, or high-road.
Yet can two cities, ferry-bound,
Have commerce over ground
Along the winding shore-line round.

And there are towns on every strand
Without or ferry or a road inland,
Which, two and two on different isles, are able
To speak through ocean by electric cable.
These triad lines must be,
Unchanged for ever, bounds of earth and sea.

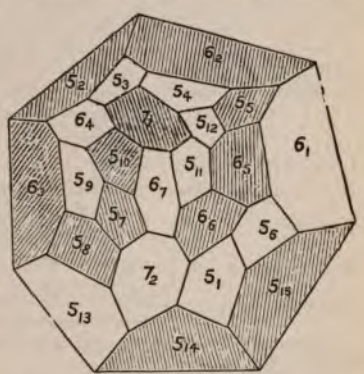
The ferries or the cable-lines between,
Some lordships of their shallow floods they mean
To bank and raise, the while they excavate
Along high road or roads, and inundate
What provinces they must, to make one strand,
And every cable a ferry, or edge of land.
How to contrive that every town may keep
Its own third line with piers along the deep?

To him who can tell them they give the remarkable profit of purchasing shares at par;
And the Council will make him an F.S.R

On islands, three, four, or two,
Towns, to threescore or two,
Cover with triedral summits your n-edron;
When they are penned, run
Over your islands a pencilling cloud,
Giving the cities the shore-lines allowed.
The white will all be
Ferry, cable, and sea.
You may feel a bit proud,
If, after some labour, you find what you want, an entire single circle of towns on a Continent.
When it is found, there is nothing to face
But proof of a rule to fit every case.

Solution by the PROPOSER.

The theorem—that on every p-edron P, having only triedral summits, a closed circle of edges passes once through every summit, has this provoking interest, that it mocks alike at doubt and proof.



Such a circle divides the face of P into a dark and a white sheet, so that no summit is unshared, *i.e.*, of three white or three dark faces, and so that there is neither a zone of white faces about a dark island, nor one of dark faces about a white lake, except the two sheets, each of which is a zone about the other. If s be the number of summits, and if A, B ... K be the m faces of the white sheet and Q, R ... V the $p-m$ dark ones, the equations following must be satisfied:

$$A+B+\ldots+K=s+2(m-1);\quad Q+R+\ldots+V=s+2(p-m-1)\ldots(F).$$

Every such selection of m faces of P that we make is a faultless white sheet, and the remaining faces a faultless dark one, if neither contains a zone nor a summit unshared.

In seeking this circle on any p-edron P, we first choose, from a promising case of equations (F), the principal faces of the white sheet, making the rest of such faces dark; we then add to both the faces dictated by the rule—no summits unshared. There is then a residue of pentagons and quadrilaterals to be so determined that neither sheet shall contain a zone nor an unshared summit. Of triangles, which can add no difficulty, I need not speak. The white zones which we run the risk of making are such as contain only white faces selected; and the like of the dark ones.

The faces collateral with the A-gon A are the zone about A; the zone about two collaterals AB is the sum of their zones, omitting A and B and repetitions. All zones are given to inspection of P, and no zone can have a summit unshared. Successes have been easy and plentiful on every P yet examined.

If P be our figure, $s = 46$, and the $p\,(=25)$ faces are $7^3 6^7 5^{15}$, where 7^3 means 777, &c. The only possible selections of white and dark sheets t

satisfy equations (F), making the base 7-gon white, and writing dark over white, are

$$\frac{6^2 5^{12}}{7^3 6^5 5^3},\quad \frac{6^5 5^8}{7^3 6^2 5^7},\quad \frac{7 6^3 5^9}{7^2 6^4 5^6},\quad \frac{7 5^{13}}{7^2 6^7 5^2},\quad \frac{7 6^6 5^5}{7^2 6^5 5^{10}},\quad \frac{7^2 6^5 5^{10}}{7 6^6 5^5},\quad \frac{7^2 6^7 5^2}{7 5^{13}},\quad \frac{7^2 6^4 5^6}{7 6^3 5^9}.$$

Sheets of all these forms will be found on some 25-edra, having the signature $7^3 6^7 5^{15}$, but not always on one P.

Starting with our cut as datum, we have two white zones about two islands $\dfrac{5_{14}5_{15}}{7_1 7_2 5_1 5_6 5_{13} 6_1}$ and $\dfrac{7_3 6_2 6_3 6_5 6_6 5_2 5_7 5_8 5_{10} 5_5}{7_1 7_2 6_1 6_4 6_7 5_1 5_3 5_4 5_6 5_9 5_{11} 5_{12} 5_{13}}$, zones which have six white faces in common, and can both be destroyed by darkening any one of the six. We may darken 5_{13}, thus making one sheet of land and one of sea, both without zones. Our sheets are

$$\frac{7_3 6_2 6_3 6_5 6_6 5_2 5_5 5_7 5_8 5_{10} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_7 5_1 5_3 5_4 5_6 5_9 5_{11} 5_{12}} \text{ of the form } \frac{7 6^4 5^8}{7^2 6^3 5^7},$$

which does not satisfy equations (F). But we see that, by making white any dark 6-gon, and darkening any white 5-gon, we can obtain the third of the above written forms. We cannot make 6_2 or 6_6 white without the unshared summit $7_1 6_1 6_2$ or $7_2 6_6 6_7$. Make 6_5 white, and 5_6 dark, to avoid the unshared summit $6_5 6_1 5_6$. We have now two sheets

$$\frac{7_3 6_2 6_3 6_6 5_2 5_5 5_6 5_7 5_8 5_{10} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_9 5_{11} 5_{12}},$$

numerically correct and containing no zone; but tactically wrong by having the unshared summits, $6_3 5_8 5_{13}$ all dark, and $6_5 5_{11} 5_{12}$ all white

Exchange 5_8 and 5_{11}; then $\dfrac{7_3 6_2 6_3 6_6 5_2 5_5 5_6 5_7 5_{10} 5_{11} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_8 5_9 5_{12}}$

is numerically and tactically a faultless pair, and if we exchange 5_8 instead of $5_8 5_{11}$ $\dfrac{7_3 6_2 6_3 6_6 5_5 5_6 5_7 5_{10} 5_{12} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_8 5_9 5_{11}}$ is another.

If the reader shades the dark sheet of either, he will see the convolutions of the white one.

Of the form $7^2 6^4 5^6$ we can select $3 \cdot 35 \cdot 5005 = 525525$ different white sheets numerically correct. We have just proved that two at least of this number give a faultless pair; and there are more than two, as given below. Let it be given that on every p-edron P having only triedral summits, this circle can be found. How far does this go towards proof that such a circle can be found on every such $(p+1)$-edron Q? Q becomes a P by erasure of an edge, and P becomes Q by drawing that edge.

Draw in our given P an edge bisecting 6_1 and making the new summits 855 and 755. The signature of this Q is $8^7 3^6 5^{17}$. Keeping the base 8-gon white, the possible sheets, by equations (F), will be of the forms

$$\frac{6^4 5^{10}}{8^7 3^6 5^7},\quad \frac{7 6^5 5^7}{8^7 5^{10}},\quad \frac{7 6^2 5^{11}}{8^7 6^3 5^6},\quad \frac{7^2 5^{12}}{8 7^6 5^5},\quad \frac{7^2 6^3 5^8}{8 7^6 2^5 9},\quad \frac{7^3 6^5 9}{8 6^4 5^8},\quad \frac{7^3 6^4 5^5}{8 6^5 1^2}.$$

The entire number of selections of every form of sheets on P is 1320396; that full number on Q is 1430618, or 110222 more. Our added risks of error on Q to those on P are the two new summits (755) and (855) to be excluded if unshared from our sheets, and the two new zones about the new 5-gons. Every other zone on Q is also a zone on P.

That is, we have in our power 110222 more selections on Q than on P; and only four more risks of error. Ergo, say I, it is impossible to doubt that we shall find on Q more faultless sheets than on P.

This is the only attempt to prove the general theorem that I desire now to offer. Demonstration it is not; but it will probably carry conviction until a p-edron is produced, on which the sought circle cannot be found. A rigorous demonstration is much to be desired, flowing from the simple definition of a p-edron with triedral summits. But I share the opinion of Professor P. G. TAIT, that our prospect of obtaining such demonstration is very remote indeed. It is my impression that he knows more about these circles than any other.

Have we here a glimpse of the Tactic Calculus which will one day make short work of questions like this? Is it impossible that the following theorem may appear on the pillars of Science? Let T_n be a given tactical success defined by the integer n, and let T_{n-1}, T_{n-2}, &c., be all given certainties. If it can be proved that the chances against the success T_{n+1} diminish without limit as n increases, T_n is to be considered a certainty for every integer n. I do not here endorse this theorem.

When the summits of a p-edron P are not all triedral, the equations (F) will be satisfied by white and dark sheets whose common circuit is a circle through all the summits; and if they are satisfied so that there is no zone, and so that every summit is shared in two contiguous white and contiguous dark portions, the required circle is found.

The equations (F) are impossible when the number s of summits is odd, while the faces are all even-angled. Such cases are easily made by crowning at m of its summits, by an m-ace making m even-angled faces, the $2r$-gonal face of a prism, completing a solid of $4r+1$ summits with even faces.

I find on our figured 25-edron one circle of the first form $\frac{6^2 5^{12}}{7^3 6^5 5^3}$, 17 of the form $\frac{6^5 5^8}{7^3 6^2 5^7}$, 8 of the form $\frac{7 6^3 5^9}{7^2 6^4 5^6}$, 4 of the form $\frac{7^2 6^5 5^{10}}{7 6^6 5^5}$, and 14 of the form $\frac{7^2 6^4 5^6}{7 6^3 5^9}$; in all 44, in which the base 7-gon is white. As in all of them the dark and white sheets can be exchanged, our puzzled planetarians have the choice of at least 88 different continents.

Attention to these circles was perhaps first called by me in *Phil. Trans.*, Vol. 148, 1858, page 160, from which the late Royal Astronomer, Sir W. R. HAMILTON, took his idea of his "Icosian Game," as he informed me, when he did me the honour to present me with his handsomest copy of the puzzle.

If neither confuting case nor rigorous demonstration of our theorem can be found, the proper thing will be to establish a conservation. Conservations are stamps of certainty which science prints on propositions about the future, which she is utterly unable to prove, but which she is led to believe by all experience on record. Such is the primary conservation which NEWTON so wisely formulated with the deliberate anthropomorphism *perseverare*, and which it is now the fashion to demonstrate scientifically by the grand law of the eternal standstill of motion. See SPENCER'S mistranslation of NEWTON, "must persevere," F. Pr. 1880, p. 182. Such are all our conservations, down to that wise upstoring some forty years ago, and its most logical consequence the present conservation, of what is poetically called the potential energy of your grandfather's chimney-pots. They are useful disguises, arithmetically harmless, of our scientific ignorance and incompetence to pronounce upon events of the future.

Why should not Tactic have its Conservations, as well as the good old Dynamic and its modern false *propylæum* Kinetic? (False in division and

in naming; first, because there is no force not kinetical; and secondly, because there is no motion not dynamical. *Vide* NEWTON'S *VIS inertiæ*. To NEWTON they appeal; to NEWTON let them go.) And why should not I be immortal for manufacturing the first? I mean to make it a dead certainty, sealed in orthodox fashion with—must be so; I know—that whenever in the 24th century these circles are looked for on the 10^{10}-edra, they will be found. For me it is quite easy—thus: There is in the essential nature of all p-edra having only triedral summits a Tactical Conservation by cyclical correlation and coordination, of perichoretical co-inter-environment and interconfiguration, due to the conformativities and circularitivities of continuous and discontinuous differentiations and integrations which we Unfathomables perfectly understand. "That is an ultimate of ultimates." Let no fathomable touch it. I made it, because I will have the applause of the gifted minds whose superior "idiosyncrasies" (*vide* Appendix to SPENCER'S *First Principles*, edition of 1880) enable them at sight exactly "to frame the ideas answering to" any of these plumed horse-marines of philosophy, "highly abstract words," even to the words of the very highest degree of abstractness "in which," after the pattern of Tactical Conservation, "Evolution at large is expressed."

Does the curious reader ask why I call "highly abstract words" horse-marines? Simply, because there are no horse-marines outside the brains of unfathomable Profundity. Is *negro* a highly black word? Is *change* or *vicissitudinality* a highly abstract one? The general idea is abstract enough.

6658. (By Prof. CROFTON, F.R.S.)—Solve the functional equations

$$\phi(ax+b) = 1+\phi(x), \qquad \phi(x^r) = 1+\phi(x), \qquad \phi\left(\frac{1}{1-x}\right) = 1+\phi(x).$$

Solution by Prof. TANNER, M.A.; Prof. MOREL; *and others.*

The equations are included in $\phi ax = 1+\phi x \ldots (1)$, where a is a given function. The solution is found by eliminating t (which must be a positive integer) from the equations $\phi x = t+k$, $x = a^t . h \ldots (2)$, where h, k are constants, connected by the relation $\phi h = k$.

One method of getting this solution is the following:—From (1) we have $\phi a^t . x = t+\phi x$; hence, writing h for x, and k for ϕh, $\phi a^t h = t+k$; showing that, if $x = a^t h$, then $\phi x = t+k$. The results are

$$\phi x = c + \frac{\log[(a-1)x+b]}{\log a}, \quad \phi x = c + \frac{\log\log x}{\log r},$$

$$\phi x = c - \frac{\log(1+\rho^2 x) - \log(1+\rho x)}{\log \rho}, \quad (\rho^2+\rho+1=0).$$

In each case c is an arbitrary constant depending on h, k.

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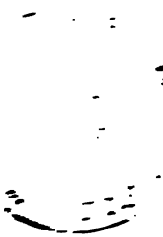

*** If the entire Twenty-six Volumes have now been published, and nearly completing a century to the papers and solutions that have appeared in the *Ladies' Diary*, about the same number of new articles and remaining contributions in all branches of Mathematics, from some of the leading Mathematicians in this and other countries.

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Solved Questions.

1843. (The Editor.)—(1) Three points being taken at random within a circle, prove that the chance that the circle circumscribing the triangle formed by joining them lies wholly within the given circle, is $\frac{2}{7}$; and (2) four points being taken at random within a sphere, prove that the chance that the sphere circumscribing the tetrahedron formed by joining them lies wholly within the given sphere, is $\frac{48\pi^2}{1925}$ ..... Page 94

2706. (J. Griffiths, M.A.)—Prove that (1) two real equilateral hyperbolas can be drawn to touch the sides of a given obtuse-angled triangle, and to pass through the centre of its circumscribed circle; and (2) the centres of these curves are the points of intersection of the nine-point and circumscribed circles of the triangle 44

5405. (S. Tebay, B.A.)—A straight rod, resting on a smooth horizontal plane, is struck by a random shot from a given point in the plane; find the mean angular velocity of the rod, and the mean motion of the centre of gravity of the rod. 53

5527. (W. S. B. Woolhouse, F.R.A.S.)—Determine, by a simple construction, the size of the equilateral triangle from or into which a triangle can be orthogonally projected. 64

5546. (The Editor.)—In a triangle ABH, right-angled at B and having AB = BH, take HC = $\frac{1}{3}$HA, and join BC; and let O be the orthocentre of the triangle ABC, DEF its orthocentric triangle, and a', b', c', ρ, Δ' the sides and inscribed radius of this orthocentric triangle, the usual notation referring to the parts of the triangle ABC; then prove that (1) tan A = 1, tan B = 2, tan C = 3; (2) $\Delta = 5\Delta' = b^2 - a^2 = \frac{1}{3}c^2$; (3) AF = 2FB, BD = $\frac{2}{3}$DC, CE = $\frac{1}{3}$EA; (4) AO = 5OD, BO = 2OE, CO = OF; (5) $\rho = \frac{1}{5}$R; and (6) develop other properties of the system. 40

5592. (J. Dawson.)—Given one point in the circumference of a circle, and two points whose respective distances from the centre are the m^{th} and n^{th} parts of the radius; show how to draw the circle. 49

5651. (Prof. Seitz, M.A.)—If a circle be drawn on the line joining two points taken at random in the surface of a given semicircle; show that the chance that the circle lies wholly within the semicircle is $\frac{4}{3} - \frac{128}{45\pi}$. ... 50

5773. (J. L. McKenzie, B.A.) — Through any point P on a circular cubic, draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P..... 88

5925. (F. T. Symons, M.A.)—If A′, B′, C′ be any three points on the edges OA, OB, OC of a tetrahedron; prove that the cosine of the angle between the planes ABC, A′B′C′ is

$$\{\ldots A, \ldots\} \cos A' - \ldots\} \div \Delta\Delta',$$

where Δ is the area of the triangle ABC, ..., and A′ the angle between the planes B′C′O, ... 121

5936. (W. H. H. Hudson, M.A.)—A sphere touches another sphere internally, and the centre of gravity of the mass included between them. If the radius of the former increase till it ultimately coincides with the latter, find the ultimate position of the centre of gravity of the included mass. Also prove that if the included matter be distributed over the surface of a sphere, that the centre of gravity may coincide with the one just found? 65

5939. (R. A. Roberts, M.A.)—If the motion of a material particle constrained to move on a given sphere, and attracted according to the law of the inverse fifth power of the distance by a uniform circular plate whose circumference is on the sphere, if the velocity in any position be that from infinity, and the direction of the force, prove that the orbit will be a circle cutting the circumference of the plate. 90

5948. (J. Rosenthal, M.A.)—Find, for the biquadratic whose roots are $\alpha, \beta, \gamma, \ldots$, the equation whose roots are $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$, &c. 87

5950. (J. L. McKenzie, B.A.) — Prove the following constructions to find the chord of curvature at a given point P on a circular cubic: — (1) Take any point A on the cubic, and draw a line through A parallel to the real asymptote, cutting the cubic in B. Let the tangent at P cut the cubic in C; draw AP cutting the cubic in D, CD in E, and BE in Q. Then Q is the point in which the circle of curvature at P cuts the cubic again. (2) Let the real asymptote cut the cubic in F; a line through P parallel to the asymptotes, in G, and the tangent at P in C. Draw CG cutting the cubic in H, and HF in Q. Then, as before, PQ is the chord of curvature at P. 88

5978. (D. Edwardes.)—If 3 balls A, B, C, of equal mass and size, moving with the same velocity V in direction s inclined at 120° to one another, impinge so that their centres form an equilateral triangle at the moment of impact; and if the coefficient of restitution between C and A or B be e, and between A and B be e'; show that A and B separate with a velocity $\frac{1}{3}(2e' + e)\,V\sqrt{3}$, it being assumed that compression ends at the same instant for all three balls. 30

6013. (Prof. Minchin, M.A.)—If at any point in a body subject to stress the principal stress consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $\sqrt{AC}$, and find the plane on which it is exerted. If the principal stresses are a tension of intensity A and two pressures of intensities B and C (B > C), show that the maximum intensity of shearing stress is $\sqrt{AB}$, and find the plane on which it is exerted. 82

6026. (W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T. 67

6029. (W. J. C. Sharp, M.A.)—Prove that the covariant

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta$$

is the locus of points whose polar conics are parabolas, and separates the points whose polar conics are ellipses from those whose polar conics are hyperbolas. 48

6040. (Prof. H. W. Lloyd Tanner, M.A.)—If

$$b^{n/a} \equiv b\,(b-a)\,(b-2a)\,\ldots\,[b-(n-1)\,.\,a],$$

prove that
$$\begin{vmatrix} b^{1/b}, & 0, & 0, & \ldots, & b^{n/a} \\ (2b)^{1/b}, & (2b)^{2/b}, & 0, & \ldots, & (2b)^{n/a} \\ (3b)^{1/b}, & (3b)^{2/b}, & (3b)^{3/b}, & \ldots, & (3b)^{n/a} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (nb)^{1/b} & (nb)^{n/b}, & (nb)^{3/b}, & \ldots, & (nb)^{n/a} - (nb)^{n/b} \end{vmatrix} = 0 \ldots\ldots 83$$

6059. (W. J. C. Sharp, M.A.)—If tangents be drawn to each of the cubics $\lambda U + \mu H$, from its point of intersection with the line $\alpha x + \beta y + \gamma z = 0$, prove that the points of contact all lie on the same quartic..... 84

6078. (Edwyn Anthony, M.A.) — Prove that (1) the principal normal at any point of a helix passes through and is perpendicular to the axis of the cylinder on which the helix is traced; (2) the angle between the binormal and the axis equals the complement of the angle between the axis and the tangent. 104

6092. (J. Young, B.A.) — Through a point in the base of a triangle produced draw a straight line cutting the sides so that the rectangle contained by the segment of one side towards the base and the segment of the other towards the vertex of the triangle shall be a maximum. 106, 115

6112. (F. Morley, B.A.) — The lengths of parallel straight lines intercepted between the angles of a tetrahedron and the opposite faces are p, q, r, s, reckoned of different signs as they fall on a face or a face produced; prove that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 0$, and deduce the corresponding theorem for a triangle. 101

6114. (E. W. Symons, B.A.)—Prove that, in a spheric triangle,

(i.) $\sec^2 A + \sec^2 B + \sec^2 C > 12$, (ii.) $\sec A \sec B \sec C > 8$;

and in a plane triangle, the triangles not being obtuse,

(i.) $\tan A + \tan B + \tan C = \tan A \tan B \tan C > 3\sqrt{3}$,
(ii.) $\cot A + \cot B + \cot C > \sqrt{3}$,
(iii.) $\tan B \tan C + \tan C \tan A + \tan A \tan B > 9$. 120

6171. (Rev. C. Taylor, D.D.)—Prove (1) that the trapezium bounded by a pair of tangents to an ellipse and the diameters to their points of contact is equal to the triangle whose sides are equal to the major axis and the focal distances of the point of concourse of the tangents; and (2) deduce therefrom solutions of Questions 2010 and 3099. 28

6175. (Rev. W. Roberts, M.A.)—P, Q, R are three points on an equilateral hyperbola whose centre is O, such that OQ bisects the angle POR; M is the middle point of the chord PQ, and N the middle point of the chord QR; express the ratio $\frac{OM \cdot PQ}{ON \cdot QR}$ in terms of the ratio $\frac{OP}{OR}$... 83

6179. (R. Knowles, B.A., L.C.P.) — If a point P be taken on the asymptote to an hyperbola whose centre is C and focus S, and if PT be drawn a tangent to the curve, prove that $\angle TSP = SPC$. 102

6185. (E. W. Symons, B.A.)—The vertex of a parabola moves along the pedal of a given curve, while its focus is fixed at the pole; prove that its envelop is the first negative pedal of the given curve..... 91

6193. (Prof. Seitz, M.A.)—Two equal small circles are drawn so as to intersect on the surface of a sphere of radius r; show that the average area of the spheric surface common to the two equal segments cut from the sphere is $(3\pi - 8)r^2$. 84

6201. (Christine Ladd.)—AB is the vertical diameter of a circle; a ball descends down the chord AC, and, being reflected by the plane BC, describes its path as a projectile; find the average range of the ball on the diameter CD, supposing all coefficients of friction relative to the descent of the ball on the chord to exist for which motion is possible. ...34

6240. (C. Leudesdorf, M.A.)—A homogeneous cube, whose edge is $2a$, strikes with one of its angular points against a perfectly rough inelastic wall. Just before impact the cube was moving in a given direction with velocity v, and rotating about an axis through its centre parallel to this direction with angular velocity ω. Prove that, if the centre of the cube begins, after the impact, to move with velocity v' in a direction making an angle θ with its direction before impact, then

$$\cos\theta = \frac{33rv'}{4a^2\omega^2 + 27v^2}. \quad \text{..... 47}$$

6242. (Prof. Rosanes.) — Prove that (1) in the plane of three given conics A, B, C there are three sets of points a, b, c, such that A_b, B_c, C_a are respectively identical with B_a, C_b, A_c; where, generally, P_q denotes the polar of a point q with respect to a conic P; (2) of the three triangles formed respectively by the three points a, the three points b, and the three points c, the first is in perspective with the second, the second with the third, and the third with the first; (3) nine points having the last-named property being given, three conics, related to them in the manner above described, are uniquely determinable. 123

6253. (B. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle..... 63

6257. (A. Buchheim, Ph.D.)—Show that

$$\iiint \frac{\sum_1^3 x_i^6 + 3\sum_1^3\sum_1^3 x_i^2 x_j^4 + 2x_1^2x_2^2x_3^2}{\left(\sum_1^3 x_i^2\right)^6}\, dx_1\, dx_2\, dx_3 = \frac{4\pi}{3abc};$$

the integration extending over the whole space outside the pedal of the ellipsoid of semiaxes a, b, c with regard to its centre the origin of coordinates. 75

6262. (Rev. H. G. Day, M.A.)—If a line of length c is divided into n parts, $x, y, z, \ldots$, by $(n-1)$ points taken at random in it, prove that the average values (1) of xyz, (2) of $x^\alpha y^\beta z^\gamma$ are

$$\frac{(n-1)!}{(2n-1)!}c^n \text{ and } \frac{(n-1)!\,\alpha!\,\beta!\,\gamma!\ldots}{(n-1+\alpha+\beta+\gamma+\ldots)!}c^{\alpha+\beta+\gamma+\ldots}. \text{ ..... } 52$$

6270. (J. L. McKenzie, B.A.) — Express in terms of the coefficients of two quadratic equations, the condition that one root of the first should have a given ratio to one root of the second. 106

6272. (E. Anthony, M.A.)—The normal at any point P of a curve intersects two straight lines, which meet in the point O, and which are at right angles to one another, in the points G and g, so that we have $m \cdot OG^2 \cdot Pg = n \cdot Og^2 \cdot PG$; find the equation to the curve. 49

6274. (Prof. Rosanes.)—The coordinates $g_1, g_2, g_3, g_4, g_5, g_6$ of a line G being given, those of its conjugate G′, relative to a surface F of the second order, are determined by six equations of the form

$$\rho g'_i = \sum_{s=0}^{s=6} C_{i,s} \cdot g_s \quad (i = 1, 2, \ldots 6),$$

where the $C_{i,s}$ denote minors of the 2nd degree, of the determinant of F, and ρ a constant factor. According to this, the determination of a *self*-conjugate line leads to a sextic equation in ρ. How is this to be reconciled with the fact that every generator of F is self-conjugate? 123

6284. (Prof. Cochez.) — Soit $f(x) = 0$ une équation algébrique, α une quelconque de ses racines. Dans l'équation $\phi(y) = 0$ dont les racines ont avec x la relation $y = f'(x)$, (1) le terme du premier degré manque; (2) le terme tout connu est le dernier terme de l'équation aux carrés des differences. 102

6287. (The Editor.)—The base AB of a triangle ABC being given in position and magnitude, and the side AC in magnitude only; trace the locus of the centre of the inscribed circle (1) generally, and (2) when AB = AC; also (3) find when the inscribed circle is a maximum, showing that then its radius is twice the distance of the third side of the triangle from the centre of the circumscribed circle; and state (4) what are the loci of the *escribed* circles. 35

6289. (G. J. Griffiths, M.A.) — A uniform heavy string of length $2a$ is placed on a smooth cardioid $r = a(1 + \cos\theta)$, whose axis is horizontal; one end of the string being at the apse; and the string is allowed to run off the curve; prove that its velocity v when just leaving the curve is given by the equation $v^2 = \frac{1}{10}ga(52 - 3^{\frac{3}{2}})$. 108

6338. (W. H. H. Hudson, M.A.)—From a semicircle, whose diameter is in the surface of a fluid, a circle is cut out, whose diameter is the vertical radius of the semicircle, find the centre of pressure of the remainder... 45

6345. (Prof. Casey, F.R.S.)—A uniform circular plate is placed with its centre upon a prop; find at what points on its circumference three given weights must be attached in order that it may rest in a horizontal position. 49

6352. (W. H. Walenn, Mem. Phys. Soc.) — Check the calculation

$$V' = \frac{1}{3} \cdot \frac{600 \times 20 - 384 \times 1}{20 - 16} \times 36 = 17568$$

by casting out the elevens, the form being preserved, and no quotients of the divisor 11 being known or used in the process. 122

6354. (J. J. Walker, M.A.)—If the coefficients of the binary quartic $(abcde)(x, y)^4$ are connected by the relations $ad - 3abc + 2b^3 = 0$, $be^2 - 3cde + 2d^3 = 0$, and if $ae - 2bd$ does not vanish, prove that the quartic is a perfect square. 30

6373. (Prof. Sylvester, F.R.S., D.C.L.)—If

$$fx = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} \ldots + L$$

$$= ax^n + nbx^{n-1} + n \cdot \tfrac{1}{2}(n-1)\, cx^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\, dx^{n-3} \ldots + l,$$

prove that (1) fx cannot have more real roots than there are continuations of sign in the series $A^2 : B^2 - AC : C^2 - BD : \ldots : L^2$; (2) fx cannot have more real roots than there are continuations of sign in the series

$$b^2 : c^2 - \tfrac{2}{1}bd : d^2 - \tfrac{3}{2}ce : e^2 - \tfrac{2}{3}\tfrac{4}{3}df : f^2 - \tfrac{3}{4}\tfrac{5}{3}eg : \ldots : l^2. \ldots 39$$

6395. (E. W. Symons, M.A.)—Three normals being drawn to an ellipse from a point in its evolute; prove (1) that the locus of the centre of the circle passing through their feet is $4(a^2x^2 + b^2y^2)^3 = a^2b^2(a^2 - b^2)^2 x^2y^2$; and (2) that one of the common chords of the ellipse and circle passes through the centre of the ellipse. 77

6398. (R. E. Riley, B.A.)—Prove that if

$(a+b+c)^3 = a^3 + b^3 + c^3$, then $(a+b+c)^{2n+1} = a^{2n+1} + b^{2n+1} + c^{2n+1}$.... 105

6403. (D. Edwardes.)—If ABCD be a quadrilateral inscribed in a circle, and H, K, L, M are the orthocentres of the triangles formed by its sides and diagonals; prove that, (1) HKLM is a quadrilateral equal and similar to ABCD; (2) A, B, C, D are the orthocentres of the triangles formed from the sides and diagonals of HKLM; (3) the lines joining similar angular points meet in a point and mutually bisect one another; and (4) that this point bisects also the line joining the centres of the circles ABCD and HKLM..... 89

6406. (Prof. Townsend, F.R.S.)—The equation of a conic in terms of the three perpendiculars λ, μ, ν on a variable tangent from the three vertices A, B, C of a fixed triangle self-reciprocal with respect to the curve, being given in the form $l\lambda^2 + m\mu^2 + n\nu^2 = 0$, where l, m, n are constants; find, in terms of the same coordinates, that of the point-pair in which the axes of the curve intersect the line at infinity. 56

6411. (Prof. Wolstenholme, M.A.)—If

$$y = x^n - nx^{n-2} + \frac{n(n-3)}{2}\, x^{n-4} - \frac{n(n-4)(n-5)}{3!}\, x^{n-6} + \ldots,$$

where n is an odd integer $(2r+1)$, prove that $\dfrac{y-2}{x-2} = (u + u_{r-1})^2$,

u_r being $x^r - (r-1)\, x^{r-2} + \dfrac{(r-2)(r-3)}{2!}\, x^{r-4} - \ldots$ 65

6423. (Rev. T. W. Openshaw, M.A.)—If a pair of conjugate diameters CP, CD of an ellipse be produced to meet the auxiliary circle in Q, Q', and the ordinates QN, Q'N' meet the ellipse in R, R'; prove (1) that

tangents at R, R′ intersect on the director-circle; and (2) extend this also to the ellipsoid; also, if the ordinates of P and D meet the auxiliary circle at K, K′, prove (3) that the tangents to the auxiliary circle at K, K′ meet on the auxiliary circle of the ellipse on which the tangents at P and D intersect. 46

6424. (J. W. Russell, M.A.)—In any triangle, prove that

$$\frac{2}{3}\sqrt{3} > (\sin A + \sin B + \sin C)^2 > 6\sqrt{3} \sin A \sin B \sin C \text{..... } 122$$

6427. (R. A. Roberts, M.A.)—Show that the cubics whose equations in rectangular coordinates are

$$(a-b)\,xy^2 + g\,(y^2-x^2) - 2fxy - k^2\,(ax+g) = 0,$$
$$(a-b)\,x^2y + f\,(y^2-x^2) + 2gxy + k^2\,(by+f) = 0,$$

cut each other at right angles at their seven finite points of intersection. 43

6430. (J. W. Sharp, M.A.)—If α, β, γ, δ, ρ be any five vectors, prove that the right part of the following expression vanishes,

$$\alpha\beta S\gamma\delta\rho + \beta\gamma S\alpha\delta\rho + \gamma\alpha S\beta\delta\rho + \gamma\delta S\alpha\beta\rho + \alpha\delta S\beta\gamma\rho + \beta\delta S\gamma\alpha\rho \text{..... } 29$$

6431. (E. W. Symons, M.A.)—Prove that the locus of a point the normals from which to a given ellipse form a harmonic pencil, is

$$(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2 \cdot c^4x^2y^2 = 0 \quad (c^2 \equiv a^2 - b^2). \text{..... } 75$$

6432. (J. Hammond, M.A.) — Prove that $PG = 2\rho$, if PG be the normal measured from the curve to the axis of x, and ρ the radius of curvature of the curve

$$x = \frac{c}{2^{\frac{1}{2}}} \int \frac{d\theta}{(1 - \frac{1}{2}\sin^2\theta)^{\frac{1}{2}}} - c\sqrt{2} \int (1 - \tfrac{1}{2}\sin^2\theta)^{\frac{1}{2}}\, d\theta, \quad y = c\cos\theta \text{... } 107$$

6446. (Prof. Genese, M.A.)—If $u \equiv (a, b, c, f, g, h)\ (x, y, 1)^2 = 0$ be the equation to a conic, prove that (1) the *directrices* are determined by the equation $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, where λ is the root of the equation $\lambda^2 - (a+b)\lambda + ab - h^2$; and (2) that, if α, β be the coordinates of the centre, the equation to the axes is

$$\frac{du}{dx} : \frac{du}{dy} = x - \alpha : y - \beta. \text{..... } 70$$

6453. (J. L. McKenzie, B.A.) — Three consecutive numbers may readily be found, each of which contains a square factor > 1; *e.g.*, 48, 49, 50; 98, 99, 100; 124, 125, 126; &c. Is the same possible for four consecutive numbers? If so, find the first case in which it occurs..... 48

6459. (C. Leudesdorf, M.A.)—A homogenous sphere of mass m and radius a is rotating about a diameter with angular velocity Ω while its centre is moving in the direction of this diameter with velocity V, when it strikes a perfectly rough horizontal plane. Show that the kinetic energy lost by the impact is $\frac{1}{2}m\mathrm{V}^2\,(1-e^2)\cos^2\theta + \frac{1}{4}m\,(\mathrm{V}^2 + a^2\Omega^2)\sin^2\theta$, θ being the angle of incidence, and e the coefficient of restitution. 54

6465. (E. W. Symons, M.A.)—Prove that the line joining the point, where $ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$ is cut by $a'x^2 + 2h'xy + b'y^2 = 0$ is

$$\frac{(\lambda a' - a)}{g}x + \frac{(\lambda b' - b)}{f}y = 2, \text{ wherein } \lambda \equiv \frac{af^2 + bg^2 - 2fgh}{a'f^2 + b'g^2 - 2fgh'}. \text{..... } 76$$

6352. (W. H. Walenn, Mem. Phys. Soc.) — Check the calculation

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$$= ax^n + nbx^{n-1} + n \cdot \tfrac{1}{2}(n-1)\, cx^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\, dx^{n-3} \ldots + l,$$

prove that (1) fx cannot have more real roots than there are continuations of sign in the series $A^2 : B^2 - AC : C^2 - BD : \ldots : L^2$; (2) fx cannot have more real roots than there are continuations of sign in the series

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6411. (Prof. Wolstenholme, M.A.)—If

$$y = x^n - nx^{n-2} + \frac{n(n-3)}{2} x^{n-4} - \frac{n(n-4)(n-5)}{3!} x^{n-6} \ldots$$

an odd integer $(2r+1)$, prove that $\frac{y-?}{x-}$

$$x^r - (r-1)\, x^{r-2} + \frac{(r-2)(r-3}{2!}$$

W. Openshaw, M.A.)—If
es be produced to meet
)N, Q'N' meet the ell

6469. (Prof. Sylvester, F.R.S.)—Let q be any integer containing only n distinct prime factors p, p_2, $p_3 \ldots p_n$, and let the n quantities 2^i-1, $2^{i_2}-1$, 2^{i_3}, $\ldots$ $2^{i_n}-1$ (where i, i_2, $\ldots$ i_n are any n integers) also only contain among them the same n prime factors; show that a rule may be given for extracting the square root of any number by means of continual multiplication or division and extraction of square root only. Apply this principle to giving rules for the extraction of all roots up to the 20th by means of the above named processes. 24

6470. (Prof. Cayley, F.R.S.)—It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form $xy-z^2 = \{(x^2-x^2)(x^2-k^2x^2)\}^{\frac{1}{2}}$. 21

6475. (The late Prof. Clifford, F.R.S.) — Let U, V be any two cubic functions of x; show that a quartic function $f(x)$ may always be found such that, by the substitution $y = U : V$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $M\,dy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant. 121

6486. (J. R. Wilson, M.A.)—From any point P on a central conic lines are drawn to the extremities of the axis BB′. BQ perpendicular to BP meets PB′ in Q, and B′R perpendicular to B′P meets PB in R. Show that the envelope of QR is a central conic, the ratio of whose axis is $a^2+b^2 : 2ab$. 47

6492. (J. Hammond, M.A.)—Prove that the sum of the infinite series

$$\frac{1}{m}+\frac{1}{m(m+1)}+\frac{1}{m(m+1)(m+2)}+\ldots = \int_0^1 e^x(1-x)^{m-1}\,dx. \ \ldots\ 35$$

6500. (G. F. Walker, M.A.)—A chord PQ of the parabola $y^2 = 4ax$ passes through a fixed point $(c, 0)$ on the axes. Circles are described touching the parabola at P and Q, and passing through the focus; show that their second common point must lie on the curve

$$(x^2+y^2-ax)^2-3c(x-a)(x^2+y^2-ax)-\frac{c^2}{a}(3a-c)y^2 = 0 \ldots\ldots 92$$

6502. (S. Tebay, B.A.) — The letters in the annexed square represent any 16 numbers in arithmetical progression, the sum of the two extremes being s. They are so arranged that $2s$ is the sum of each of the 24 groups *abcd*, *efgh*, *ijkl*, *mnpq*, *aeim*, *bfjn*, *cgkp*, *ahlq*, *afkq*, *dgjm*, *abef*, *bcgf*, *cdgh*, *efij*, *fgjk*, *ghkl*, *klpq*, *ijmn*, *jknp*, *acik*, *bdjl*, *egmp*, *fhnq*, *admq*. Find in how many essentially different ways the numbers can be thus arranged, exclusive of all reversions, such as *dcba*, *miea*, &c.; and also the total number of ways in which four of these numbers make up $2s$. 57

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*l*
m	*n*	*p*	*q*

6507. (Prof. Wolstenholme, M.A.)—If $a_1, a_2 \ldots, b_1, b_2 \ldots$ be all positive quantities, prove that

$$\int_0^\infty \frac{\sin a_1x \sin a_2x \ldots \sin a_nx \cos b_1x \cos b_2x \ldots \cos b_mx \sin \lambda x}{x^{n+1}}\,dx$$

will be $\frac{1}{2}\pi a_1a_2 \ldots a_n$, if λ have any value not less than $\Sigma(a)+\Sigma(b)$..... 86

6517. (J. J. Walker, M.A.)—The points of intersection of two conics $ax^2 \ldots = 0$, $a'x^2 \ldots = 0$ are given by

$$(C_1y^2-2F_1yz+B_1z^2)^2-4(Cy^2-2Fyz+Bz^2)(C'y^2\ldots) = 0,$$

and two similar equations; $C_1 \ldots$, $C \ldots$, $C' \ldots$ being coefficients of the contravariant conics of the system, viz., $C_1 = ab'+a'b-2ff' \ldots$. 23

6519. (W. S. McCay, M.A.)—If two tangents be drawn to a nodal cubic from a point on the curve, prove that the line joining the points of contact envelopes a conic having triple contact with the cubic; and show that, if the conic be $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, the conic is

$$5(x^2+y^2+z^2+22(yz+zx+xy) = 0. \quad \ldots\ldots\ldots 29$$

6528. (R. B. Hayward, M.A.)—In a "three-cornered" constituency (*i.e.*, one which returns three members) each voter has two votes, but cannot give both to the same candidate. Supposing the majority, consisting of M voters, to put forward three candidates, and the minority, consisting of m voters, to put forward two, and supposing that all the voters take part in the election and give both their votes; prove that (1) the chance that the three candidates of the majority will be all elected is $1-3\frac{(m+1)(m+2)}{(M+1)(M+2)}$ or $\frac{(2M-3m-1)(2M-3m-2)}{(M+1)(M+2)}$, according as m is $<$ or $>$ $\frac{1}{3}$M, and $=\frac{1}{4}$ nearly if $m=\frac{1}{3}$M; and (2) that, if $m=\frac{2}{3}$M, the chance that two of the three will be defeated is $\frac{1}{3}$ nearly..... 55

6529. (E. W. Symons, M.A.)—From a point four normals are drawn to a conic; prove that, if the line joining two of their feet move parallel to itself, the line joining the other two will move parallel to itself; and find the locus of the point in order that this may be possible. 75

6536. (Prof. Ball, F.R.S.)—If k be the constant term in the equation of a surface, and $\Delta = 0$ the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x\frac{d}{dk}}\Delta = 0$? 28

6555. (H. Stewart, B.A.)—If

$$\frac{bc(y+z)-a^2x}{a(b^2+c^2-bc)} = \frac{ca(z+x)-b^2y}{b(c^2+a^2-ca)} = \frac{ab(x+y)-c^2z}{c(a^2+b^2-ab)},$$

prove that $(a^2x+b^2y+c^2z)(b^2c^2+c^2a^2+a^2b^2) = 3a^2b^2c^2(x+y+z)$..... 76

6559. (R. Rawson.)—If $E(x, y) = (x-y_1)\ldots(x-y_n)$, and $\psi(x, y)$ be any function of (x, y), $f(x)$ being rational in x, and of lower dimension than n; show that (1) the integral

$$\int\left\{\frac{f(y_1)\psi(y_1\cdot y)y_1^m}{\frac{dF(y_1\cdot y)}{dy_1}}+\ldots\frac{f(y_n)\psi(y_n\cdot y)y_n^m}{\frac{dF(y_n\cdot y)}{dy_n}}\right\}dy+C$$

is the coefficient of $\frac{1}{x^{m+1}}$ in $f(x)\int\frac{\psi(x, v)}{F(x, y)}dy$; and (2) show the application of this property to prove the celebrated theorem of Abel in ultra-elliptic functions. 31

6564. (Prof. Townsend, F.R.S.)—A point, taken arbitrarily in the plane of a uniform circular lamina attracting inversely as the square of the distance, and its reflexion with respect to the centre, being supposed the foci, if internal of an ellipse, and if external of an hyperbola, having the containing diameter for transverse axis of figure; show that the potential of the attraction for either point is equal to the mass per unit of area of the lamina, multiplied into the circumference of the ellipse in the former case, and into the difference between the circumference of the hyperbola and the sum of its asymptotes in the latter case..... 77

6570. (Prof. Genese, M.A.)—The angle which a fixed diameter of a rectangular hyperbola subtends at a variable point is divided into two parts, whose sines are in a given ratio; prove that the dividing line passes through a fixed point. 43

6571. (C. B. S. Cavallin, M.A.) — Taking the variation of the Croftonian integral $\iint (\theta - \sin\theta)\, dx\, dy = \frac{1}{4}L^2 - \pi\Omega$, (*Phil. Trans.*, 1868, p. 188), on the supposition that the contour of reference (of length L inclosing an area Ω) changes to another nearly situated, generated by prolonging each radius ρ of curvature in Ω a length $\mu f(\rho)$, where μ is an infinitely small constant, prove that we obtain

$$\iint \left\{ \frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right\} \sin^2 \tfrac{1}{2}\theta \, dx\, dy = \tfrac{1}{4} L \int_0^{2\pi} f(\rho)\, d\phi,$$

where t_1, t_2 are the lengths of the tangents, drawn from the point (x, y) to Ω; ρ_1, ρ_2 the radii of curvature at the corresponding points of contact; ρ the radius of curvature at an arbitrary point of the curve; and ϕ the inclination of this radius to a fixed line in the plane; the integration extending over the whole plane outside Ω. 40

6572. (The Editor.)—If from the ends of a diameter AB of a circle, tangents AP, BP be drawn to a circle that touches the given circle and its diameter AB, prove that the sum or the difference of AP and BP will be equal to the sum or the difference of AB and the perpendicular PQ thereon from P, according as the contact of the tangential circle with the given circle and its diameter AB is internal or external. 62, 110

6574. (W. B. Grove, B.A.)—Suppose a series of 73 cards to be painted with red, blue, yellow, and green, every card but one receiving at least one colour. Let it be observed that 21 have *some* part coloured red, 48 blue, 31 yellow, and 46 green; also 14 have both red and blue upon them, 16 both red and green, 14 both blue and yellow, 28 both blue and green, 20 both yellow and green, and 9 have all four colours. Also 15 are painted with blue *alone*, 6 with yellow alone, but none with either red or green alone. Find the laws (designed or accidental) according to which the colours are arranged. 27, 72

6575. (H. McColl, B.A.)—Let S denote any multiple integral; αS the value of S when the integration is restricted by the statement α; $(\alpha - \beta + \gamma - \ldots)$ S an abbreviation for $\alpha S - \beta S + \gamma S - \ldots$; and $\alpha(1-\beta)(1-\gamma)$S an abbreviation for $(\alpha - \alpha\beta - \alpha\gamma + \alpha\beta\gamma)$ S, and so on; then show that

$$\alpha\beta'\gamma' \ldots S = \alpha(1-\beta)(1-\gamma) \ldots S. \quad \ldots\ldots 55$$

6577. (Rev. T. R. Terry, M.A.)—If a triangle PQR be inscribed in a parabola, so that PQ, PR are the normals at Q, R; show that (1) the centre of gravity of the triangle lies on the axis of the parabola; and (2) the side QR passes through a fixed point on the axis produced at a distance from the vertex equal to the semi-latus rectum. 51

6583. (W. J. C. Sharp, M.A.)—Show that the equation

$$\left(\frac{1+p^2}{q}\frac{d}{dx}\right)^n \cdot \frac{(1+p^2)^3}{q} = 0 \text{ represents any } (n-1)^{\text{th}} \text{ involute of a circle.} \quad 54$$

6589. (S. Tebay, B.A.)—The letters in the annexed square represent the first sixteen consecutive numbers, so arranged that each of the sixteen groups *abef, cdgh, ijmn, klpq, fgjk, admq, abcd, efgh, ijkl, mnpq, aeim, bfjn*

egkp, *dhlq*, *afkq*, *dgjm* makes 34. There are 432 essentially different arrangements of these numbers, but a demonstration of the following property, with other allied properties, is still a desideratum. If $a+c=17$, prove that $e+g=17$. 57

6597. (Prof. Townsend, M.A.)—A uniform flexible chord, in free equilibrium under the action of a central repulsive force varying inversely as the square of a distance, being supposed to have throughout its entire extent the tension to infinity under the action of the force; determine, for any assigned positions of its two terminal points with respect to the centre of force, the requisite length of the cord and the appropriate form of the catenary. 67

6598. (C. B. S. Cavallin, M.A.)—Prove that, (1) by starting from the Croftonian integral $\iint \theta\, dx\, dy = \pi\Theta - \int_0^{2\pi} \Sigma d\omega$, (*Phil. Trans.* 1868, p. 190) and only varying the *inner* contour, we get

$$\iint \left\{ \frac{f(\rho_1)}{i_1} + \frac{f(\rho_2)}{2} \right\} dx\, dy = \int_0^{2\pi} cf(\rho)\, d\omega,$$

where θ is the angle which at any point within the annulus (Θ) between an interior and exterior contour is subtended by the former; c is a chord of the latter intercepting a segment Σ of it and touching the former; ω the angle which c makes with a fixed line, and the other notation as in Quest. 6571. Also, (2) by only varying the *outer* contour, prove that we get

$$\int_0^{2\pi} (\pi - \theta)\, \rho f(\rho)\, d\phi = \int_0^{2\pi} \left\{ \int_0^{\beta} \rho f(\rho)\, d\phi \right\} d\omega,$$

where ρ is now the radius of curvature of the outer contour at the point at which the inner subtends an angle θ; β the angle between the normals at the ends of c, and the rest of the notation as in (6571), but the integration extends only over the annulus between the two contours..... 41

6601. (Prof. Wolstenholme, M.A.)—Prove that any quadric surface $u=0$ is its own polar reciprocal with respect to any surface whose equation is $2u u_0 = \left(x_0 \frac{du}{dx} + y_0 \frac{du}{dy} + z_0 \frac{du}{dz} + w_0 \frac{du}{dw} \right)^2$, u being a homogeneous function of x, y, z, w, and u_0 the same function of x_0, y_0, z_0, w_0. 25

6611. (Rev. T. R. Terry, M.A. Suggested by Question 6590.)—If there be any distribution of mass M in space, and I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, A and B, respectively; prove that the moment of inertia about a parallel line through the centre of gravity of the mass is

$$\frac{a^2 I_a^2 + \ldots - (a^2+b^2-c^2) I_a I_b - \ldots - a^2(b^2+c^2-a^2) M I_a - \ldots + M^2 a^2 b^2 c^2}{M(a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2)}.$$ 34

6612. (E. B. Elliott, M.A.)—A spherical surface of radius a, made of thin perfectly flexible material whose internal side is reflecting, is deformed without either crumpling or stretching, and assumes the form of a surface of revolution. From a point Q on the axis of this surface, and within it, a small pencil of light is incident on it at any point A. Show that, if q_1, q_2 be the primary and secondary foci of the reflected pencil, and if R be the image of Q by reflection in the tangent plane at A,

$$Rq_1 \cdot Rq_2 : Aq_1 \cdot Aq_2 = 4QA^2 : a^2. \ldots 74$$

6613. (G. S. Carr, B.A.)—If A, B, C are the normal, and F, G, H the tangential stresses upon a rectangular element of a strained elastic solid, show—(1) that the resultant tangential stress, at the same point, upon a plane whose direction-cosines with respect to the axes of A, B, C are l, m, n, is $\left[\left\{F(m^2-n^2)+(Gm-Hn)l-(B-C)mn\right\}^2+\&c.\right]^{\frac{1}{2}}$;
and (2) that its direction-cosines are proportional to
$l\left\{A(m^2+n^2)-Bm^2-Cn^2-2mnF\right\}+(m^2+n^2-l^2)(Gn-Hm)$, &c.... 51

6615. (H. McColl, B.A.)—Speaking of the limits of multiple integrals if z_r (the r^{th} limit of z) be less than z_m and greater than z_n, we have the statemental equation $z_{m'n}=z_{m'r}+z_{r'n}$. In what sense is this equation true when z_r is not thus restricted? 50

6627. (Prof. Townsend, F.R.S.)—Two shallow circular arcs, of arbitrary versed-sines, being supposed described on either segment, and on the same produced outwards to half its length, of a uniform elastic rod, supported in a horizontal position by one central and two terminal props, and bent slightly by its own weight; show that the vertical depression varies as the product of the ordinates to the arcs throughout the entire extent of the segment. 90

6631. (Prof. Matz, M.A.)—Show that the average area of parallelograms inscribed in a triangle is one-third of the area of the triangle. 47

6636. (Lizzie A. Kittudge.)—On a plane field, the *crack* of the rifle and the *thud* of the ball striking the target are heard at the same instant; find the locus of the hearer. 23

6640. (R. Rawson.)—Assuming that

$$u=\left\{\phi_1(x)\,\theta(x,y)+1\right\}^2-\phi_2(x)=(x-y_1)(x-y_2)\ldots(x-y_n)\ldots\ldots(1),$$

where u, $\phi_1(x)$, $\theta(x,y)$, and $\phi_2(x)$ are rational in x; show that

$$\int\frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\left[\phi_2(y_1)\right]^{\frac{1}{2}}}+\ldots\ldots\int\frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\left[\phi_2(y_n)\right]^{\frac{1}{2}}}=\frac{1}{x}\left|\frac{\epsilon\psi(x)}{\phi_1(x)\left[\phi_2(x)\right]^{\frac{1}{2}}}\right.$$
$$\times\log\frac{\phi_1(x)\,\theta(x,y)+1-\left[\phi_2(x)\right]^{\frac{1}{2}}}{\phi_1(x)\,\theta(x,y)+1+\left[\phi_2(x)\right]^{\frac{1}{2}}}\ldots\ldots(2),$$

where $\epsilon=\mp1$, $x\,|\,f(x)$ is used as a convenient symbol to represent the coefficient of x in the development of $f(x)$, and $x^{-1}\,|\,f(x)$ is the coefficient of x^{-1} in the development of $f(x)$, &c..... 91

6643. (A. Stein, Ph.D.)—A mark is made on a vertical tower at a known height from a horizontal plane, the altitude of this mark and of the top of the tower is observed from a point in the plane; find the probable error of the height of the tower deduced from these angles in terms of the probable error in the measurement of an angle; and show that the best position of observation is that in which the sum of the two altitudes is a right angle. 24

6649. (J. O'Regan.) — If a quadrilateral be circumscribed to a circle and a fifth variable tangent be drawn, the rectangles under perpendiculars on it from each pair of opposite angles are in a constant ratio. 108

6650. (D. Edwardes.)—If tangents are drawn to the circumscribed circle of a triangle ABC at the angular points, and if Δ be the area of the triangle so formed, and Δ_1 that of the orthocentric triangle of ABC, prove, with the usual notation, that $\Delta^{\frac{1}{2}}:\Delta_1^{\frac{1}{2}}=R:\rho$. 34

6651. (E. W. Symons, B.A.)—A parabola has contact of the 4th order with a given circle; prove that its focus moves on one concentric circle, and that its directrix touches another. 42

6653. (W. R. Westropp Roberts, M.A.)—A heavy elliptical lamina is placed in contact with a vertical wall OA, and with a horizontal wall OB, and moves in a vertical plane. Show that, the walls being smooth, the lamina leaves the vertical wall when

$$2(\sin\alpha-\sin\phi)\left\{1+\frac{r^4a^2b^2\sin^2\phi\cos 2\phi}{(r^4\sin^2\phi\cos^2\phi-a^2b^2)(5r^4\sin^2\phi\cos^2\phi-4a^2b^2}\right\}=\sin\phi,$$

where $r^2=a^2+b^2$, $\phi=\widehat{COB}$, C being the centre of the lamina..... 81

6655. (G. F. Walker, M.A.)—Show that, according as q is an integer numerically greater than or less than p,

$$\int_0^{\frac{1}{2}\pi}\cos^{2p}\theta\cos^2q\theta\,d\theta=\frac{\pi}{2^{2p+2}}\frac{(2p)!}{(p!)^2},$$

or

$$\frac{\pi}{2^{2p+2}}\left\{\frac{(2p)!}{(p!)^2}+\frac{(2p)!}{(p+q)!\,(p-q)!}\right\}. \text{ ..... } 52$$

6663. (Prof. Wolstenholme, M.A.)—If p, q be the lengths of two tangents drawn to an ellipse $a^2y^2+b^2x^2=a^2b^2$ from any point O of the ellipse $a^2y^2+b^2x^2=m^2a^2b^2$, prove that

(1) $$\left\{\frac{p^2+q^2}{a^2+b^2}-4\left(1-\frac{1}{m^2}\right)^2\right\}^2\left(\frac{a^2+b^2}{2ab}\right)^2=\left(\frac{2}{m^2}-1\right)^2\left(\frac{p^2q^2}{a^2b^2}-\frac{4(m^2-1)^3}{m^4}\right);$$

(2) when $m^2<(a^2+b^2)a^{-2}$, then pq will have a minimum value when the point O is at the ends of the major axis, and a maximum when at the ends of the minor; (3) when $m^2>(a^2+b^2)a^{-1}$ and $<(a^2+b^2)b^{-2}$, then pq will have maxima values when O is at any vertex and minima values (all equal) when the two tangents are at right angles; and (4) when $m^2>(a^2+b^2)b^{-2}$ the maximum values will be when O is at the ends of the major axis and the minimum when at the ends of the minor, while there are no other maxima nor minima. 21

6665. (Prof. Nash, M.A.)—If two conics have the same focus S, and touch one another at P, prove that (1) the tangent at P will pass through Z, the intersection of the directrices corresponding to S; (2) a parabola described with focus S, touching the minor axes of these conics will touch PZ at Z, and its directrix will pass through P. 29

6666. (Prof. Genese, M.A.)—If P be a point on a rectangular hyperbola, AA′ any diameter; the tangent to the curve at P, and the tangent to the circle APA′ at P, divide AA′ internally and externally in the duplicate of the ratio AP : A P. 26

6667. (The Editor.)—Given the distances of a point from three of the corners of a square, (1) construct the square; and prove (2) that if a, b, c be these distances and Δ the area of a triangle whose sides are a, $b\sqrt{2}$, c, the area of the square is $\frac{1}{2}(a^2+c^2)\pm 2\Delta$; also (3) extend the problem to the case of any rectangle or triangle given in species, showing that, in the case of a rectangle whose sides are as $m:n$ and area Σ, if Δ be the area of a triangle with sides a, b', c', where

$$b'=\frac{(m^2+n^2)^{\frac{1}{2}}}{n}b,\quad c'=\frac{m}{n}c,\quad \text{then } \Sigma=\frac{1}{m^2+n^2}\{mn(a^2+c^2)\pm 4n^2\Delta\},$$

+ or − according as the angle subtended at the point by the diagonal is obtuse or acute..... 73

6673. (Rev. W. A. Whitworth, M.A.)—Find a triangle whose altitude and sides are expressed by four consecutive integers, and show that only one such triangle exists. 42

6682. (H. G. Dawson.) — If $(n-1)$ of the roots of $(abc\dots)(x, 1)^n$ are connected by the relation $\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=0$, show that the remaining root w is given by $z=aw+b$,

$$z^3-3\frac{(n-1)^2}{n+1}.Hz+\frac{(n-2)(n-1)^3}{2(n+1)}.G=0,$$

where $H=b^2-ac$, $G=2b^3-3abc+a^2$; and hence deduce Quest. 6638. 33

6688. (C. Leudesdorf, M.A.)—If a, b, c be three quantities such that any two are together greater than the third, and if x, y, z be three quantities whose sum is positive; show that, if $a^2x^{-1}+b^2y^{-1}+c^2z^{-1}=0$, the product xyz must be negative. 56

6696. (D. Edwardes.) — If $m\sin(\theta+\phi)=\cos(\theta-\phi)$, prove that $(1-m\sin 2\theta)^{-1}+(1-m\sin 2\phi)^{-1}=2(1-m^2)^{-1}$. 53

6700. (Prof. Morel.)—On considère un cercle, un triangle inscrit ABC, et un triangle circonscrit A'B'C', tel que les points de contact du second sont aux sommets du premier. D'un point quelconque M de la circonférence, on abaisse des perpendiculaires MP, MQ, MR sur les côtés du premier triangle et des perpendiculaires MP', MQ', MR' sur les côtés du second. Démontrer que l'on a toujours

$$MP.MQ.MR=MP'.MQ'.MR'. \dots 82$$

6714. (J. J. Walker, M.A.)—Given two sides b, c of a spherical triangle ABC, of which the angle A is equal to the sum of the angles B, C; show that these angles are determined by

$$\cos A=-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c \quad \dots\dots(1),$$

$$\sin\tfrac{1}{2}b\cot B=\sin\tfrac{1}{2}c\cos^{\frac{1}{2}}\tfrac{1}{2}(b+c)\cos^{\frac{1}{2}}\tfrac{1}{2}(b-c)\dots\dots(2).$$

..... 26

6720. (T. C. Simmons, M.A.)—AB, AC are two fixed tangents to a circle [or a conic]; DE, FG are two other tangents meeting AB in D, F, and AC in E, G; prove that the point of intersection of EF and DG lies in BC. 103

6722. (D. Edwardes.)—Prove that the mean distance from its centre of all points within an oblate spheroid, of equatorial radius a, is

$$\tfrac{3}{8}(ae^{-1}\sin^{-1}e+b) \dots 71$$

6725. (W. H. H. Hudson, M.A.) — If Δ be the area, and P the perimeter, of a triangle drawn on a sphere of radius r, and if Δ', P' be the area and perimeter of the polar triangle; prove (1) that

$$\frac{\Delta}{r^2}+\frac{P'}{r}=\frac{\Delta'}{r^2}+\frac{P}{r}=2\pi;$$

and (2) state the corresponding proposition for any spherical polygon. 44

6735. (Prince Camille de Polignac.)—An unclosed polygon is inscribed in a conic and circumscribed about another; M_1, M_2 are two consecutive fixed sides; a_1, a_2 any other pair of consecutive sides taken in the same order; a_1 meets M_2 in m_2; and a_2 meets M_1 in m_1. If the line m_1m_2 passes through a fixed point, prove that the conics have double contact. 93

6738. (Prof. Crofton, F.R.S.)—Prove that

$$D^n f(xD)X=f(xD+n)D^nX. \dots 63$$

6745. (A. McIntosh, B.A.)—If in a nodal cubic, with the nodal tangents at right angles, a right-angled triangle be inscribed having the right angle at the double point; show that (1) the hypotenuse passes through a fixed point on the curve; and (2) in the case of the Folium of Descartes, this point is the vertex or apse of the loop. 38

6751. (The Editor.)—A circle of given radius is drawn at random in a given circle; find the chance that a radius drawn at random in the fixed circle will cut the other circle. 85

6763. (Rev. C. Taylor, D.D.)—If the axes of one rectangular hyperbola are parallel to the asymptotes of another, and the centre of each lies on the other; prove that an infinity of circles can be drawn through the centre of either so as to meet the other again in conjugate triads with respect to the former. 99

6765. (Prince Camille de Polignac.) — Find the condition for the convergence of the continued fraction $A+\frac{p}{q+}\ \frac{p}{q+}\ \frac{p}{q+}$ &c. 109

6766. (Prof. Cayley, F.R.S.)—Find the stationary and double tangents of the curve $x^4+y^4+z^4=0$. 64

6770. (Prof. Wolstenholme, M.A.) — A parallelogram of minimum perimeter is inscribed in a given ellipse; prove that a conic can be found which cuts the ellipse at right angles at each of the angular points of the parallelogram, and that this conic passes through four fixed points (two real and two impossible)..... 118

6776. (W. H. H. Hudson, M.A.)—A uniform rod, of length $2a\sin\alpha$, is placed within a rough vertical circle, of radius a, and is on the point of motion, the coefficient of friction at its upper and lower ends being $\tan\lambda'$, $\tan\lambda$; prove that, if θ be the inclination to the vertical of the line joining the centre of the sphere to the centre of the rod,

$$\tan\theta=\frac{\sin(\lambda+\lambda')}{2\cos(\alpha+\lambda)\cos(\alpha-\lambda')};$$

and examine the case where $\alpha+\lambda=\frac{1}{2}\pi$. 85

6778. (J. J. Walker, M.A.)—Show that the work done (in *C.G.'s*) in raising the piston of a suction pump, so as to elevate the water b *cms* in the suction-tube (of section a) from a point a *cms* below the bottom of the working-barrel, is equal to

$$a\left\{\tfrac{1}{2}b^2-a\mathrm{H}\log_e[\mathrm{H}\div(\mathrm{H}-b)]+\mathrm{H}ab\div(\mathrm{H}-b)\right\},$$

H being the height of the water-barometer; and explain the significance of the separate terms..... 115

6779. (Rev. A. J. C. Allen, B.A.)—A uniform elastic rod is cut into three equal portions, of height h, and these are placed upright on a horizontal plane, the three being in the same vertical plane and at equal distances (l) from each other. A heavy uniform elastic rod, of length $2l$ and weight w, is placed on the top of them; prove that the pressures on the middle and either of the outside supports are

$$\frac{5}{8}w\left(1+\frac{24}{5}\cdot\frac{\mathrm{K}h}{\mathrm{E}l^3}\right)\div\left(1+9\cdot\frac{\mathrm{K}h}{\mathrm{E}l^3}\right),\quad \frac{3}{16}w\left(1+16\cdot\frac{\mathrm{K}h}{\mathrm{E}l^3}\right)\div\left(1+9\cdot\frac{\mathrm{K}h}{\mathrm{E}l^3}\right),$$

where K and E are the Young's modulus of the supports, and the flexural rigidity of the beam. 98

6780. (R Rawson.)—Boole has given (*Diff. Eqs.*, p. 459) the theorem from Curtis (*Camb. Math. Jour.*, Vol. ix., p. 280)

$$\frac{d^2u}{dx^2} + 2Q\frac{du}{dx} + \left\{ Q^2 + \frac{dQ}{dx} \pm c^2 - \frac{m(m+1)}{x^2} \right\} u = 0,$$

which can be integrated in finite terms when Q is any function of x. Show that the more general equation

$$\frac{d^2u}{dx^2} + 2Q\frac{du}{dx} + \left\{ Q^2 + \frac{dQ}{dx} + Ax^r + \frac{(m-r)(m+r+4)}{4(m+2)^2 x^2} \right\} u = 0$$

can be integrated in finite terms, where A is any constant, and $(2p+1)\,m = -4p$; p being any integer..... 68

6783. (R. Knowles, B.A., L.C.P.) — Prove that (1) if the axes be rectangular, the equation to the locus of the vertices of all parabolas whose chords of contact cut off a triangle of constant area ($\frac{1}{2}a^2$), is $(x^7y^{-1})^{\frac{1}{12}} + (y^7x^{-1})^{\frac{1}{12}} = a^{\frac{1}{4}}$; (2) that of the foci is $(x^2+y^2)^2 = a^2xy$; (3) the chords of contact always touch at their middle points a rectangular hyperbola, to which the axes are asymptotes. 109

6795. (Prof. Sylvester, F.R.S.)—Prove that

$$1 + 1\,.\,m + 1\,.\,5\frac{m(m-1)}{1} + 1\,.\,5\,.\,9\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} + \ldots \text{ is divisible by } 2^m.$$

Ex.: $1 + 5 + 10\,(5) + 10\,(5\,.\,9) + 5\,(5\,.\,9\,.\,13) + 5\,.\,9\,.\,13\,.\,17$

$= 1 + 5 + 50 + 450 + 2925 + 9945 = 13376 = 2^5 \times 418$.... 97

6800. (The Editor.)—Prove that, if

$$\frac{ayz}{y^2+z^2} = \frac{bzx}{z^2+x^2} = \frac{cxy}{x^2+y^2} = 1, \text{ then } a^2+b^2+c^2 = abc+4. \ldots\ldots 106$$

6801. (C. W. Merrifield, F.R.S.)—Prove that the continued product $m^n!$, where m is prime, is divisible without remainder by m to the power of $(m^n-1) : (m-1)$, and the quotient is not again divisible by m. 66

6804. (Rev. W. A. Whitworth, M.A.) — The sides of a triangle are expressed by integers in arithmetical progression; if each side be increased by 50, the radius of the inscribed circle will be increased by 17; but, if each side be increased by 60, the radius of the inscribed circle will be increased by 20; find the triangle..... 101

6807. (G. F. Walker, M.A.) — In any curve in which the difference between the radii of absolute and spherical curvature is constant, prove that the arcs of the loci of the centres of absolute and spherical curvature (measured between corresponding points) are equal. 104

6811. (Rev. C. Taylor, D.D.) — Focal chords of a parabola at right angles to one another meet the directrix in T, t. Show that (1) the bisectors of the angles between the tangents from either of the points T, t are parallel to the tangents from the other; and (2) every pair of the four tangents intersect at constant angles. 78

6816. (Dr. Macfarlane, F.R.S.E.)—A colony is formed by N persons of different races, and the two sexes are equally numerous. The colony remains unaffected by either immigration or emigration. What is the least time in which it can become homogeneous? 101

6826. (Prof. Sylvester, F.R.S.)—Find the greatest common measure of D_n and D_{n+1}, where D_n represents a determinant of the n^{th} order of the form

$$\begin{vmatrix} 1 & 1 & . & . & . & . \\ 1 & 3 & 2 & . & . & . \\ . & 1 & 5 & 3 & . & . \\ . & . & 1 & 7 & 4 & . \\ . & . & . & 1 & 9 & 5 \\ . & . & . & . & 1 & 11 \end{vmatrix}. \quad \ldots\ldots\ldots\ldots 97, 117$$

6829. (Prof. Wolstenholme, M.A.)—Given a fixed point S and a fixed straight line, SX is drawn perpendicular to the line and bisected in O; p being any point in the plane, pk is let fall perpendicular on the line, and kO, pS meet in P. Prove that (1) the relation between p, P is reciprocal; (2) whatever curve be traced out by p, the locus of P will be a curve of the same order and class; (3) the tangents at p, P to their loci intersect always on the fixed straight line; (4) the loci of p, P will coincide when it is a conic of which S and the given straight line are focus and corresponding directrix..... 100

6830. (Prof. Crofton, F.R.S.)—Prove that

$$e^{D+x^{-1}}\,F(x) = \frac{x+1}{x}\,F(x+1). \quad \ldots\ldots\ldots 96$$

6834. (The Editor.) — Trace the locus of a point whose distance from a given line $y+c=0$ is equal to the sum (including, throughout, *difference*) of its distances from two given points (a, b), $(-a, -b)$, showing that (1) its general equation is

$$[(y+c)^2+4(ax+by)]^2 = 4(y+c)^2[(x+a)^2+(y+b)^2];$$

(2) when $b=0$, so that the curve is the locus of the vertex of a triangle on a given base, having the sum of its two sides equal to the sum of the perpendicular from the vertex on the base and a given line (c), the equation is $(y+c)^4+16a^2x^2 = 4(y+c)^2(a^2+x^2+y^2)$; (3) when $c>2a$, the curve (2) consists of a loop and four infinite branches that cross at the intersection of the y-axis with the given line, and have an asymptote parallel to that line at a distance $2a$ on each side therefrom; (4) analogous but varying forms subsist when $2a>c>a\sqrt{3}$ and $c<a\sqrt{3}$, with a conjugate point on the y-axis when $c=a\sqrt{3}$; (5) when $c=2a$, so that, in the triangle-locus (2), the sum of the sides is equal to the sum of the base and perpendicular, the equation is $4(a^2-x^2)(y+4a) = (3y+8a)y^2$, and then two of the infinite branches degenerate into the base of the triangle, and the other two unite continuously with the loop and cut the y-axis at an angle of $48°\,12' = \cos^{-1}\frac{2}{3}$; (6) the curve (5) is also the locus of the intersection of tangents drawn from the ends of a diameter of a circle to a circle that touches the given circle and its diameter; (7) the areas of the loop of the curve (5), of the part cut off by the base, and of the space between the infinite branches and the asymptote, are, if we put $\sin^2\beta = \frac{5}{9}$,

$$\frac{2}{3^{\frac{3}{2}}}(7\sqrt{15}-16\beta)a^2, \quad \frac{16}{27}(9-\pi\sqrt{3})a^2, \quad \frac{2}{3^{\frac{3}{2}}}(7\sqrt{15}+16\beta)a^2;$$

(8) the locus of the centre of the touching circle in (6) is a parabola; and (9) the sum of the tangents of the angles subtended at each end of the given diameter in (6) by lines drawn from the other end to the centre of the touching circle, is unity. 111

6859. (Prof. Simon Newcomb, M.A.)—Prove that

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\ldots-2\eta\cos x$$

$$-\tfrac{1}{2}\cdot 2\eta^2\cos 2x-\tfrac{1}{3}\cdot 2\eta^3\cos 3x-\ldots=\Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix. \quad 116$$

6861. (Prof. Genese, M.A.)—A, B, C, D are fixed points. A circle is drawn through AB; then two can be drawn through CD to touch it; prove that the locus of the points of contact is a bicircular quartic.... 113

6882. (Bello Easton.) — Through a point P, between two lines AB, AC given in position, draw a line such that the rectangle under the parts thereof between the point and those lines may be a minimum... 115

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6470. (By Prof. CAYLEY, F.R.S.) — It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form $xy-z^2 = \{(z^2-x^2)(z^2-k^2x^2)\}^{\frac{1}{2}}$.

Solution by W. J. C. SHARP, M.A.

I have shown (*Quarterly Journal*, Vol. XVI., p. 186), that the equation to a proper cubic can always be reduced to the form

$$a\xi^3+3c_1\xi\zeta^2+c\zeta^3-3b_3\eta^2\zeta=0, \text{ that is, } \xi^3+\mu\xi\zeta^2+\zeta^3+\eta^2\zeta=0.$$

Comparing this with the proposed form,

$$xy^2-2yz^2+(1+k^2)xz^2-k^2x^3=0,$$

the transformation may be effected by putting

$$\zeta=\nu^2\{(1+k^2)x-2y\}, \quad \nu\eta=z, \quad \xi=Ax+By,$$

where the four unknowns A, B, k^2, ν are to be determined by substituting, for ξ and ζ in $(k^2x^2-y^2)x=\xi^3+\mu\xi\zeta^2+\zeta^3$, and equating coefficients.

6663. (By Prof. WOLSTENHOLME, M.A.)—If p, q be the lengths of two tangents drawn to an ellipse $a^2y^2+b^2x^2=a^2b^2$ from any point O of the ellipse $a^2y^2+b^2x^2=m^2a^2b^2$, prove that

(1) $$\left\{\frac{p^2+q^2}{a^2+b^2}-4\left(1-\frac{1}{m^2}\right)^2\right\}^2\left(\frac{a^2+b^2}{2ab}\right)^2=\left(\frac{2}{m^2}-1\right)^2\left(\frac{p^2q^2}{a^2b^2}-\frac{4(m^2-1)^3}{m^4}\right);$$

(2) when $m^2 < (a^2+b^2)a^{-2}$, then pq will have a minimum value when the point O is at the ends of the major axis, and a maximum when at the ends

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of the minor; (3) when $m^2 > (a^2+b^2)\,a^{-1}$ and $< (a^2+b^2)\,b^{-2}$, then pq will have maxima values when O is at any vertex and minima values (all equal) when the two tangents are at right angles; and (4) when $m^2 > (a^2+b^2)\,b^{-2}$ the maximum values will be when O is at the ends of the major axis and the minimum when at the ends of the minor, while there are no other maxima nor minima.

Solution by G. HEPPEL, M.A.; A. MARTIN, M.A.; *and others.*

1. Let O be $x = ma\cos\theta$, $y = mb\sin\theta$; then chord of contact is

$$mb\cos\theta\,.\,x + ma\sin\theta\,.\,y = ab,$$

and the points of contact are

$$mx = a\cos\theta \pm (m^2-1)^{\frac{1}{2}}\,a\sin\theta, \quad my = b\sin\theta \mp (m^2-1)^{\frac{1}{2}}\,b\cos\theta.$$

If we put $m = \sec\phi$, we get $x = a\cos(\theta\mp\phi)$, $y = b\sin(\theta\mp\phi)$; from which, after simplifying and reducing, we obtain

$$2p^2\cot^2\phi = a^2+b^2-(a^2-b^2)\cos 2(\theta-\phi),$$
$$2q^2\cot^2\phi = a^2+b^2-(a^2-b^2)\cos 2(\theta+\phi);$$

therefore, by adding and subtracting,

$$(a^2-b^2)\cos 2\theta = \frac{a^2+b^2-(p^2+q^2)\cot^2\phi}{\cos 2\phi}; \quad (a^2-b^2)\sin 2\theta = -\frac{(p^2-q^2)\cot^2\phi}{\sin 2\phi};$$

whence, eliminating θ, and replacing functions of ϕ by functions of m,

$$\left\{\frac{p^2+q^2}{a^2+b^2}-4\left(1-\frac{1}{m^2}\right)^2\right\}^2\left(\frac{a^2+b^2}{2ab}\right)^2 = \left(\frac{2}{m^2}-1\right)^2\left(\frac{p^2q^2}{a^2b^2}-4\,\frac{(m^2-1)^3}{m^4}\right).$$

2. We have $u = 4p^2q^2 = \tan^4\phi\left\{(a^2+b^2)^2-2(a^4-b^4)\cos 2\theta\cos 2\phi + \frac{1}{2}(a^2-b^2)^2(\cos 4\theta+\cos 4\phi)\right\}$;

therefore $$\frac{du}{d\theta} = \tan^4\phi\left\{4(a^4-b^4)\cos 2\phi\sin 2\theta - 2(a^2-b^2)^2\sin 4\theta\right\},$$

$$\frac{d^2u}{d\theta^2} = \tan^4\phi\left\{8(a^4-b^4)\cos 2\phi\cos 2\theta - 8(a^2-b^2)^2\cos 4\theta\right\}.$$

Now $\dfrac{du}{d\theta} = 0$ in three cases,—(i.) when $\theta = 0$ or π, (ii.) when $\theta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, (iii.) when $(a^2+b^2)\cos 2\phi = (a^2-b^2)\cos 2\theta$.

In case (i.) O is at the end of the major axis, and the examination of the sign of $\dfrac{d^2u}{d\theta^2}$ shows that u is a minimum from $m = 1$ to $m = (a^2+b^2)a^{-2}$, and a maximum from $m = (a^2+b^2)a^{-2}$ to $m = \infty$.

In case (ii.) O is at the end of the minor axis, and u is a maximum from $m = 1$ to $m = (a^2+b^2)b^{-2}$, and a minimum from $m = (a^2+b^2)b^{-2}$ to $m = \infty$.

In case (iii.) the sign of $\dfrac{d^2u}{d\theta^2}$ depends upon that of

$$(a^2+b^2)\cos 2\phi\cos 2\theta - (a^2-b^2)\cos 4\theta,$$

or on that of $(a^2-b^2)\left\{\cos^2 2\theta - \cos 4\theta\right\}$, or on that of $\sin^2 2\theta$; therefore there is a minimum. And in this case we have

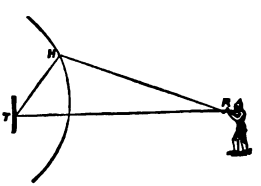
$$a^2(\cos 2\theta - \cos 2\phi) = b^2(\cos 2\theta + \cos 2\phi),$$
$$a^2 \sin(\theta+\phi)\sin(\varphi-\theta) = b^2 \cos(\theta+\phi)\cos(\theta-\phi).$$

Comparing this with the equations of the points of contact, we see that it gives the condition that the two tangents are at right angles. Lastly, as the locus of the intersection of pairs of tangents at right angles is the circle $x^2+y^2 = a^2+b^2$, we infer, first, that the limits of m to render such tangents possible are from $(a^2+b^2)a^{-2}$ to $(a^2+b^2)b^{-}$; and secondly, that all the four minima for any value of m are equal, because the tangents are drawn from points symmetrically situated.

6636. (By Lizzie A. Kittudge.)—On a plane field, the *crack* of the rifle and the *thud* of the ball striking the target are heard at the same instant; find the locus of the hearer.

Solution by C. Morgan, M.A.; W. M. Coates, B.A.; *and others.*

Let r_1, r_2 be the respective distances of the hearer (H) from the rifleman (R), and the target (T); v the velocity of sound in air (assuming that the sounds arising from a *crack* and *thud* travel through air with equal velocity); and t the time the ball takes to reach the target; then



$$\frac{r_1}{v} = \frac{r_2}{v} + t, \quad \text{or} \quad r_1 - r_2 = vt;$$

hence the locus of the hearer (H) is an hyperbola whose foci are R and T.

6517. (By J. J. Walker, M.A.)—The points of intersection of two conics $ax^2\ldots = 0$, $a'x^2\ldots = 0$ are given by,

$$(C_1y^2 - 2F_1yz + B_1z^2)^2 - 4(Cy^2 - 2Fyz + Bz^2)(C'y^2\ldots) = 0,$$

and two similar equations; $C_1\ldots$, $C\ldots$, $C'\ldots$ being coefficients of the contravariant conics of the system, viz., $C_1 = ab' + a'b - 2ff'\ldots$

Solution by the Proposer.

The invariant (I) of $S \equiv ax^2 + 2(hy+gz)x + by^2 + 2fyz + cz^2$, considered as a quadratic in x is $I = a(by^2+2fyz+cz^2) - (hy+gz)^2 = Cy^2 - 2Fyz + By^2$, where $C = ab - h^2\ldots$ Similarly the invariant (I′) of $S' \equiv a'x^2 + \ldots$ Also the invariant of the system S, S′, considered as two quadratic forms in x,

is $J = a(b'y^2+2f'yz+c'z^2)+a'(by^2+2fyz+cz^2)-2(hy+gz)(h'y+g'z)$
$= C_1y^2-2F_1yz+B_1y^2,$

where $C_1 = ab'+a'b-2hh'$... Now the resultant of the two quadratics is equal to $J^2-4\Pi'$. Hence, to determine the common $y : z$, we have the equation given in the Question.

6643. (By A. Stein, Ph.D.)—A mark is made on a vertical tower at a known height from a horizontal plane, the altitude of this mark and of the top of the tower is observed from a point in the plane: find the probable error of the height of the tower deduced from these angles in terms of the probable error in the measurement of an angle; and show that the best position of observation is that in which the sum of the two altitudes is a right angle.

Solution by Prof. Matz, M.A.; *the* Rev. T. R. Terry, M.A.; *and others.*

Let a be the height of the mark, b of the tower, and ϕ and θ the angles subtended; then $b = a\tan\theta\cot\phi$. If δb be the error consequent on an error dx in measuring an angle, supposed the same in each case,

$$b+\delta b = a\left(\tan\theta+\frac{\delta x}{\cos^2\theta}\right)\left(\cot\phi-\frac{\delta x}{\sin^2\phi}\right), \quad \therefore\ \delta b = \tfrac{1}{2}a\frac{\sin 2\phi-\sin 2\theta}{\sin^2\phi\cos^2\theta}\delta x.$$

Therefore the best position is given by $\sin 2\phi = \sin 2\theta$, or $\theta+\phi = \frac{1}{2}\pi$.

6469. (By Professor Sylvester, F.R.S.)—Let q be any integer containing only n distinct prime factors $p, p_2, p_3 \ldots p_n$, and let the n quantities 2^i-1, $2^{i_2}-1$, $2^{i_3}-1$, ... $2^{i_n}-1$ (where $i, i_2, \ldots i_n$ are any n integers) also only contain among them the same n prime factors; show that a rule may be given for extracting the square root of any number by means of continual multiplication or division and extraction of square root only. Apply this principle to giving rules for the extraction of all roots up to the 20th by means of the above named processes.

Solution by W. J. C. Sharp, M.A.

Any fraction whatever, $p : q$ say, may be expressed either as a terminating or a recurring radix fraction in the scale of 2.

If $\frac{p}{q} = \frac{1}{2^a}+\frac{1}{2^b}+\frac{1}{2^c}+$ &c., where a, b, &c. are in ascending order of magnitude, we have $n^{\frac{p}{q}} = n^{\frac{1}{2^a}} \times n^{\frac{1}{2^b}} \times n^{\frac{1}{2^c}} +$ &c.,

since the only digits in this scale are 0 and 1. Prof. Sylvester [in the annexed *Note*], gives $n^{\frac{1}{3}} = n^{\frac{1}{4}+\frac{1}{16}+}$ &c. Now $\frac{1}{3} = \cdot\dot{0}\dot{1}$ in the scale of two, which leads to the same result. His other examples may also be deduced in the same way.

As Prof. SYLVESTER anticipates, a similar statement will hold if any number a be substituted for 2; but as the digits are more numerous in other scales, involution may be required.

[In elucidation of the theorem, Prof. SYLVESTER adds the following note:—"To understand it, suppose you want the cube root of N: this is

$$N^{\frac{1}{3}} = N^{\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots} = N^{\frac{1}{3}} . N^{\frac{1}{3^3}} . N^{\frac{1}{3^4}} \ldots$$

Again, if you want the 5th root,

$$\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\ldots = \frac{1}{15}, \quad \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots = \frac{1}{3};$$

therefore $$\frac{1}{5} = \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots-2\left(\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\ldots\right).$$

Here $$N^{\frac{1}{5}} = N^{\frac{1}{4}} . N^{-\frac{1}{5^2}} . N^{\frac{1}{5^3}} . N^{-\frac{1}{5^4}} \ldots$$

So, again, $\frac{1}{3^2} = \frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\ldots$, therefore $N^{\frac{1}{3}} = N^{\frac{1}{4}} . N^{\frac{1}{4^2}} . (N^{\frac{1}{4^3}})^3 \ldots$

And so in general. I presume that it is true in general (as it certainly is for all the cases up to 20) that we may always find n prime factors (n being some finite number), and n numbers i such that 2^i-1 shall only contain the n factors in question; probably also the i's may all be taken primes. Moreover, any number, or at least any prime number a, no doubt may be substituted for 2."]

6601. (By Prof. WOLSTENHOLME, M.A.) — Prove that any quadric surface $u=0$ is its own polar reciprocal with respect to any surface whose equation is $2u u_0 = \left(x_0\frac{du}{dx}+y_0\frac{du}{dy}+z_0\frac{du}{dz}+w_0\frac{du}{dw}\right)^2$, u being a homogeneous function of x, y, z, w, and u_0 the same function of x_0, y_0, z_0, w_0.

Solution by G. F. WALKER, M.A.; CHRISTINE LADD; *and others.*

If (x', y', z', w') be any point on $u=0$, the pole of the tangent plane at (x', y', z', w') with respect to $2u_0u = \left(x_0\frac{du}{dx}\ldots\right)^2$ is given by the equations

$$\left\{-u_0\frac{du}{dx}+\left(x_0\frac{du}{dx}\ldots\right)\frac{du_0}{dx_0}\right\} \div \frac{du'}{x'} = \&c. = \&c.;$$

each of these equal quantities is easily* found to be equal to

$$\left\{-u_0x+\left(x_0\frac{du}{dx}\ldots\right)x_0\right\}\frac{1}{x'} = \&c. = \&c.,$$

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* Prof. WOLSTENHOLME remarks that, whenever he sees the word "easily" so used, he always interprets it to mean "with some difficulty."

and therefore the required polar reciprocal is given by

$$\left\{-u_0\frac{du}{dx}+\left(x_0\frac{du}{dx}\cdots\right)\frac{du_0}{dx_0}\right\}\left\{-u_0x+\left(x_0\frac{du}{dx}\cdots\right)x_0\right\}\cdots=0,$$

$$2u_0^2u-u_0\left(x_0\frac{du}{dx}\cdots\right)2\left(x_0\frac{du}{dx}\cdots\right)+\left(x_0\frac{du}{dx}\cdots\right)^2 2u_0=0,$$

which is $u=0$.

6666. (By Professor Genese, M.A.)—If P be a point on a rectangular hyperbola, AA′ any diameter; the tangent to the curve at P, and the tangent to the circle APA′ at P, divide AA′ internally and externally in the duplicate of the ratio AP : A′P.

Solution by R. Knowles, B.A., L.C.P.; G. Heppel, M.A.; *and others.*

Take the ordinary axes, let the point P be (h, k), and let the tangents to the hyperbola and circle cut the major axis in T, Q respectively. Let $CT=x_1$, $CQ=x_2$. Then the equation to the tangent to the hyperbola is $hx-ky=a^2$, hence $CT=\frac{a^2}{h}$. Also $QP^2=QA.QA'$, $(x_2-h)^2+k^2=x_2^2-a^2$; and from the equation to the hyperbola $k^2=h^2-a^2$; therefore $x_2=h$;

hence we have $$\frac{AT}{A_1T}=\frac{ah+a^2}{ah-a^2}=\frac{h+a}{h-a},\quad \frac{AQ}{A_1Q}=\frac{h+a}{h-a},$$

and $$\frac{AP^2}{A_1P^2}=\frac{k^2+(h+a)^2}{k^2+(h-a)^2}=\frac{2h^2+2ah}{2h^2-2ah}=\frac{h+a}{h-a}.$$

[The tangent to the circle at P is perpendicular to the major axis.]

6714. (By J. J. Walker, M.A.)—Given two sides b, c of a spherical triangle ABC, of which the angle A is equal to the sum of the angles B, C; show that these angles are determined by

$$\cos A=-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c \quad\ldots\ldots\ldots\ldots(1),$$

$$\sin\tfrac{1}{2}b\cot B=\sin\tfrac{1}{2}c\cos^2\tfrac{1}{2}(b+c)\cos^2\tfrac{1}{2}(b-c) \quad\ldots\ldots\ldots\ldots(2).$$

Solution by R. Tucker, M.A.; Prof. Matz, M.A.; *and others.*

Let D be the mid-point of BC, and take $AD=p=BD=CD$; then

$$\tan\tfrac{1}{2}c=\tan p\cos B,\ \tan\tfrac{1}{2}b=\tan p\cos C \quad\ldots\ldots\ldots\ldots(\alpha).$$

Now $\cos B\cos C-\sin B\sin C=\cos A=-\cos B\cos C+\sin B\sin C\cos 2p$, whence $\cos A=-\cos B\cos C\tan^2 p=-\tan\frac{1}{2}b\tan\frac{1}{2}c$, by (α).

Or thus, by Napier's Analogies,

$$\tan \tfrac{1}{2}A = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2}(B+C) = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2}A;$$

therefore $\tan^2 \frac{1}{2}A = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)}$, whence $\cos A = -\tan \frac{1}{2}b \tan \frac{1}{2}c$.

Again, $\cos b + \cos c = 2\cos^2 p$, $\therefore \cos p = \cos \frac{1}{2}(b+c) \cos \frac{1}{2}(b-c)$...($\beta$);

$\cos p = \cot B \cot \frac{1}{2}q$, $\cos p = \cot C \tan \frac{1}{2}q$, if $\angle ADB = q$,

therefore $\cos^2 p = \cot B \cot C$.....(γ);

$$\cot B \tan p = \frac{\tan \frac{1}{2}c \sin c}{\sin q \sin p}, \quad \cot C \tan p = \frac{\tan \frac{1}{2}b \sin b}{\sin q \sin p},$$

therefore $$\frac{\cot B}{\cot C} = \frac{\sin^2 \frac{1}{2}c}{\sin^2 \frac{1}{2}b} \text{.....}(\delta).$$

From (γ), (δ), $\cot^2 B = \cos^2 p \frac{\sin^2 \frac{1}{2}c}{\sin^2 \frac{1}{2}b}$; hence, remembering ($\beta$), we have

$$\cot B \sin \tfrac{1}{2}b = \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(b+c) \cos \tfrac{1}{2}(b-c).$$

6574. (By W. B. Grove, B.A.)—Suppose a series of 73 cards to be painted with red, blue, yellow, and green, every card but one receiving at least one colour. Let it be observed that 21 have *some* part coloured red, 48 blue, 31 yellow, and 46 green: also 14 have both red and blue upon them, 16 both red and green, 14 both blue and yellow, 28 both blue and green, 20 both yellow and green, and 9 have all four colours. Also 15 are painted with blue *alone*, 6 with yellow alone, but none with either red or green alone. Find the laws (designed or accidental) according to which the colours are arranged. [This question is related to what Prof. Jevons calls the inverse logical problem, which Mr. Grove states that he has completely solved by Mr. McColl's logical notation.]

Solution by Dr. Macfarlane, F.R.S.E.

Let C denote the collection of cards, r red, b blue, y yellow, g green. Then the data are

$$C = 73, \quad C(1-r)(1-b)(1-y)(1-g) = 1, \quad Cr = 21, \quad Cb = 48,$$
$$Cy = 31, \quad Cg = 46, \quad Crb = 14, \quad Crg = 16, \quad Cby = 14,$$
$$Cbg = 28, \quad Cyg = 20, \quad Crbyg = 9, \quad C\overline{b}(1-r)(1-y)(1-g) = 15,$$
$$Cy(1-r)(1-b)(1-g) = 6, \quad Cr(1-b)(1-y)(1-g) = 0,$$
$$Cg(1-r)(1-b)(1-y) = 0.$$

From these equations I first obtain, by the elementary processes of the *Algebra of Logic*, the values of the other non-negative terms, namely, $Cry = 14$, $Crby = 14$, $Crbg = 9$, $Cryg = 9$, $Cbyg = 9$. By expanding the terms, I obtain the following laws:—

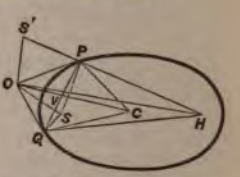
$$Crb(1-y) = 0, \quad Cry(1-b) = 0, \quad Cby(1-r) = 0 \text{.....} (1, 2, 3),$$
$$Cr(1-g)(1-b)(1-y) = 0, \quad Cg(1-r)(1-b)(1-y) = 0 \text{.....} (4, 5).$$

This set of laws evidently contains no redundancy.

The cards which are red and blue are yellow; those which are red and yellow are blue; those which are blue and yellow are red; those which are red, and not blue and not yellow, are green; those which are green, and not blue and not yellow, are red. [Or thus:—the cards that are red must be either both blue and yellow, or, if neither, green, and *vice versâ*.]

6171. (By the Rev. C. Taylor, D.D.)—Prove (1) that the trapezium bounded by a pair of tangents to an ellipse and the diameters to their points of contact is equal to the triangle whose sides are equal to the major axis and the focal distances of the point of concourse of the tangents; and (2) deduce therefrom solutions of Questions 2010 and 3099.

Solution by D. Edwardes; H. Hillary, M.A.; *and others.*



Let OP, OQ be the tangents, S, H, C the foci and centre. Then, evidently,

$$2\Delta PCQ = \Delta PSQ + \Delta PHQ;$$

hence $2\Delta OPQ + 2\Delta PCQ = OPSQ + OPHQ$.

Since the perpendiculars from O on SP, SQ, HP, HQ are equal, this gives $2OPCQ = p.AA'$, p being the length of these perpendiculars, and AA′ the major axis. If HP be produced to S′, so that HS′ = AA′, then OS = OS′ and $\frac{1}{2}p.AA' =$ area of OHS′; hence area OPCQ = area OHS′. [From this theorem we have, in Quest. 2010 (*Reprint*, Vol. VII, p. 43) $AQ.AR \sin A + \ldots + \ldots = 2\Delta ABC$, &c.

Again, if CO meet PQ in V, we have, in Quest. 3099 (Vol. XIV, p. 74),

$$\Delta OHS' = OPQ + CPQ = OPQ\left(1 + \frac{CV}{VO}\right); \quad \therefore \frac{OP.OQ}{OS'.OH} = \frac{\Delta OPQ}{\Delta OHS'} = \frac{OV}{OC}.]$$

6536. (By Professor Ball, F.R.S.)—If k be the constant term in the equation of a surface, and $\Delta = 0$ the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x\frac{d}{dk}}\Delta = 0$?

Solution by W. J. C. Sharp, M.A.

By the symbolical form of Taylor's Theorem, $e^{-x\frac{d}{dk}}\Delta(k) \equiv \Delta(k-x)$; and, if $U = k$ be the original surface, $U = k - x$ will be the equation to a surface which passes through the intersection of the other three, and is similar and similarly situated to $U = k$. Hence the roots determine these surfaces.

6430. (By J. W. SHARPE, M.A.)—If α, β, γ, δ, ρ be any five vectors, prove that the right part of the following expression vanishes,

$$\alpha\beta S\gamma\delta\rho + \beta\gamma S\alpha\delta\rho + \gamma\alpha S\beta\delta\rho + \gamma\delta S\alpha\beta\rho + \alpha\delta S\beta\gamma\rho + \beta\delta S\gamma\alpha\rho$$

Solution by the PROPOSER; S. RUGGERO; *and others.*

The right part of the expression is

$$\begin{aligned}
&V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho + V\alpha(-\gamma S\beta\delta\rho + \delta S\beta\gamma\rho) + V\beta(\gamma S\alpha\delta\rho + \delta S\gamma\alpha\rho)\\
&= V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho + V\alpha(-\sigma S\gamma\beta\delta + \beta S\delta\gamma\rho) + V\beta(\rho S\gamma\alpha\delta - \alpha S\delta\gamma\rho)\\
&= -V\rho\{\alpha S\beta\gamma\delta - \beta S\gamma\delta\alpha\} - V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho\\
&= -V\rho\{V(V\gamma\delta \,.\, V\beta\alpha)\} - V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho\\
&= -V(V\mu\lambda - \rho) - \lambda S\mu\rho + \mu S\lambda\rho = -(\mu S\lambda\rho - \lambda S\mu\rho) - \lambda S\mu\rho + \mu S\lambda\rho = 0,
\end{aligned}$$

where λ and μ are put for $V\alpha\beta$, $V\gamma\delta$, respectively.

6519. (By W. S. MCCAY, M.A.)—If two tangents be drawn to a nodal cubic from a point on the curve, prove that the line joining the points of contact envelopes a conic having triple contact with the cubic; and show that, if the conic be $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0$, the conic is

$$5(x^2 + y^2 + z^2) + 22(yz + zx + xy) = 0.$$

Solution by W. H. BLYTHE, B.A.; Prof. NASH, M.A.; *and others.*

Taking θ^3, $(1-\theta)^3$, -1 to be the coordinates of any point on the cubic, (SALMON'S *Higher Curves*, p. 181), the equation of the tangent at (θ) is $(1-\theta)^2 x + \theta^2 y + \theta^2(1-\theta)^2 z = 0$; hence, if φ_1, φ_2 be the points of contact of tangents from θ, ϕ_1, ϕ_2 are roots of the equation $\phi^2 + 2\phi(\theta - 1) - \theta = 0$; therefore the equation of the line joining ϕ_1, ϕ_2 is

$$x(4\phi^2 - \phi + 1) + y(4\phi^2 - 7\phi + 4) + z(\phi^2 - \phi + 4) = 0;$$

and the envelop of this is

$$(x + 7y + z)^2 = 4(4x + 4y + z)(x + 4y + z),$$

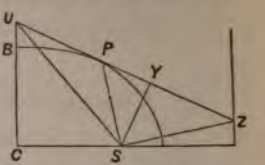
or

$$5(x^2 + y^2 + z^2) + 22(yz + zx + xy) = 0.$$

This conic touches the cubic at the points $(-8, 1, 1)$, $(1, -8, 1)$, $(1, 1, -8)$, which are the points of contact of the tangents drawn to the cubic from the points of inflexion.

6665. (By Professor NASH, M.A.)—If two conics have the same focus S, and touch one another at P, prove that (1) the tangent at P will pass through Z, the intersection of the directrices corresponding to S; (2) a parabola described with focus S, touching the minor axes of these conics will touch PZ at Z, and its directrix will pass through P.

Solution by DOMENICO MONTESANO; G. HEPPEL, M.A.; *and others.*



Let CA, CB be the semi-axes, S the focus, of a conic, and let SY be perpendicular to the tangent at P which meets the directrix in Z and the minor axis in U. Then, because PSZ is a right angle, $PZ \cdot PY = SP^2$.

Therefore, (1) if the positions of the tangent, point of contact, and focus are given, PZ is constant, and the directrices of all the conics pass through Z.

Also, (2) if a parabola touch PZ in Z, its directrix must pass through P, because ZSP is a right angle. And it is easy to show that the triangles CSU, PSY are similar, whence $\angle CUS = \angle PSY = \angle PZS$, and therefore UC produced must be a tangent to the parabola. Hence, by the rule of identity, parabolas touching the minor axes of the conics must pass through Z. [This theorem is the reciprocal of the inverse of Euc. III. 11, 12, the centres of inversion and reciprocation being the same.]

6354. (By J. J. WALKER, M.A.)—If the coefficients of the binary quartic $(abcde)(xy)^4$ are connected by the relations $a^2d - 3abc + 2b^3 = 0$, $be^2 - 3cde + 2d^3 = 0$, and if $ae - 2bd$ does not vanish, prove that the quartic is a perfect square.

Solution by the PROPOSER.

Multiplying the conditions by de, ab respectively, and subtracting, $a^2d^2e - 2abd^3 - ab^2e^2 + 2b^3de = 0$, or $(ae - 2bd)(ad^2 - b^2e) = 0$; consequently, if $ae - 2bd$ does not vanish, $a : e = b^2 : d^2$. Again, from the second condition, $\frac{3c}{e} = \frac{b}{d} + \frac{2d^2}{e^2}$. Hence $(a \ldots e)(xy)^4$ may be transformed into

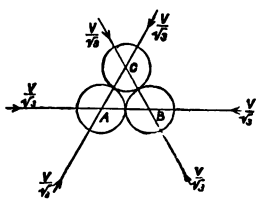
$$e\left\{\frac{b^2}{d^2}x^4 + 4\frac{b}{d}x^3y + \left(2\frac{b}{d} + 4\frac{d^2}{e^2}\right)x^2y^2 + 4\frac{d}{e}xy^3 + y^4\right\},$$

which is equal to $\quad e\left(\frac{b}{d}x^2 + 2\frac{d}{e}xy + y^2\right)^2$.

5978. (By D. EDWARDES.)—If 3 balls A, B, C, of equal mass and size, moving with the same velocity V in direction s inclined at 120° to one another, impinge so that their centres form an equilateral triangle at the moment of impact; and if the coefficient of restitution between C and A or B be e, and between A and B be e'; show that A and B separate with a velocity $\frac{1}{3}(2e' + e)\,V\sqrt{3}$, it being assumed that compression ends at the same instant for all three balls.

Solution by Professor EVANS, M.A.; E. RUTTER; *and others.*

The V of A before impact is the resultant of two velocities $\frac{1}{2}V\sqrt{3}$ in directions AB, AC. If u, u' be the velocities of A, B after impact in AB, we have, by the third law of motion,



$$\tfrac{1}{2}mV\sqrt{3}-\tfrac{1}{2}mV\sqrt{3} = mu+mu',$$

therefore $u' = -u$;

also $u-u' = -\frac{1}{2}e'(V+V)\sqrt{3}$,

or $u = -\frac{1}{2}e'V\sqrt{3}$.

Similarly for the impact between A and C, if v be velocity of A after impact in AC; $v = -\frac{1}{2}eV\sqrt{3}$. The relative velocity of A to B after impact is evidently in direction AB, and the velocity of A in this direction is

$$-\tfrac{1}{2}e'V\sqrt{3}-\tfrac{1}{2}eV\sqrt{3}\cos 60° = -\tfrac{1}{4}(2e'+e)V\sqrt{3}.$$

Hence relative velocity

$$= \tfrac{1}{4}\sqrt{3}\,[(2e'+e)V+(2e'+e)V] = \tfrac{1}{2}(2e'+e)V\sqrt{3}.$$

6559. (By R. RAWSON.) — If $E(x, y) = (x-y_1)\dots(x-y_n)$, and $\psi(x, y)$ be any function of (x, y), $f(x)$ being rational in x, and of lower dimension than n; show that (1) the integral

$$\int\left\{\frac{f(y_1)\,\psi(y_1\cdot y)\,y_1^m}{\frac{dF(y_1\cdot y)}{dy_1}}+\dots\frac{f(y_n)\,\psi(y_n\cdot y)\,y_n^m}{\frac{dF(y_n\cdot y)}{dy_n}}\right\}dy+C$$

is the coefficient of $\frac{1}{x^{m+1}}$ in $f(x)\int\frac{\psi(x, y)}{F(x, y)}dy$; and (2) show the application of this property to prove the celebrated theorem of Abel in ultra-elliptic functions.

Solution by the PROPOSER.

Since $F(x, y) = x-y_1\,.\,x-y_2\,.\,x-y_3\dots x-y_n$; then, by the usual theorem in the decomposition of rational fractions,

$$\frac{f(x)}{F(x, y)} = \Sigma\cdot\frac{f(y_n)}{(x-y_n)\frac{dF(y_n\cdot y)}{dy_n}} = \Sigma\cdot\frac{f(y_n)}{\frac{dF(y_n\cdot y)}{dy_n}}\left\{\frac{1}{x}+\frac{y_n}{x^2}+\frac{y_n^2}{x^3}+\dots\frac{y_n^m}{x^{m+1}}\right\}$$

$$= \frac{s}{x}+\frac{s_1}{x^2}+\dots\frac{s_m}{x^{m+1}}+\frac{s_{m+1}}{x^{m+2}}+\dots\frac{s_{2m}}{x^{2m+1}} \quad\dots\dots(1),$$

where

$$s_m = \Sigma\cdot\frac{f(y_n)\,y_n}{\frac{dF(y_n\cdot y)}{dy_n}} \quad\dots\dots(2).$$

Multiply (1) successively by $a, a_1x, a_2x^2, a_3x^3, \ldots a_mx^m$, where a, a_1, a_2, &c. are functions of y; then

$$\left.\begin{aligned}
\frac{af(x)}{F(x,y)} &= \frac{as}{x} + \ldots \frac{as_m}{x^{m+1}} + \frac{as_{m+1}}{x^{m+2}} + \ldots \frac{as_{2m}}{x^{2m+1}} \\
\frac{a_1xf(x)}{F(x,y)} &= a_1s + \ldots \frac{a_1s_m}{x^m} + \frac{a_1s_{m+1}}{x^{m+1}} + \ldots \frac{a_1s_{2m}}{x^{2m}} \\
&\vdots \\
\frac{a_mx^mf(x)}{F(x,y)} &= a_ms x^{m-1} + \ldots \frac{a_ms_m}{x} + \frac{a_ms_{m+1}}{x^2} + \ldots \frac{a_ms^{2m}}{x^{m+1}}
\end{aligned}\right\} \quad \ldots\ldots(3).$$

From (3), we have

$$\text{coef. of } \frac{1}{x^{m+1}} \text{ in } (a + a_1x + \ldots a_mx^m)\,\frac{f(x)}{F(x,y)} = as_m + a_1s_{m+1} + \ldots a_ms_{2m} \ldots(4).$$

Restoring the values of s_m, s_{m+1}, &c., as given by (2),

$$\text{coefficient of } \frac{1}{x^{m+1}} \text{ in } (a + a_1x + \ldots a_mx^m)\,\frac{f(x)}{F(x,y)} = \Sigma \cdot \frac{f(y_n)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,(a + a_1y_n + \ldots a_my_n^m) \quad \ldots\ldots(5).$$

If, therefore, $\psi(x, y) = a + a_1x + \ldots a_mx^m$,

we have $\psi(y_n \cdot y) = a + a_1y_n + \ldots a_my_n^m$.

Substituting in (5),

$$\text{coefficient of } \frac{1}{x^{m+1}} \text{ in } \frac{f(x)\,\psi(x,y)}{F(x,y)} = \Sigma \cdot \frac{f(y_n)\,\psi(y_n \cdot y)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}} \quad \ldots\ldots(6),$$

Integrating with respect to y,

$$\text{coef. of } \frac{1}{x^{m+1}} \text{ in } f(x) \int \frac{\psi(x,y)}{F(x,y)}\,dy = \Sigma \cdot \int \frac{f(y_n)\,\psi(y_n \cdot y)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy + c \ldots(7).$$

Now, Abel has shown (pp. 282—299 of his "Math. Works," by Holmboe) that the sum of the (n) integrals

$$\pm \Sigma \cdot \int \frac{(fy_n)\,dy_n}{(y_n - \alpha)[\phi(y_n)]^{\frac{1}{2}}} = \Sigma \cdot \int \frac{\lambda_1(y_n \cdot y)}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy - \int \frac{\lambda(\alpha, y)}{F(\alpha, y)}\,dy + C \quad \ldots(8),$$

where $\lambda(x, y) = 2f(x)\left\{\theta(x, y)\,\dfrac{d\theta_1(x,y)}{dy} - \theta_1(x, y)\,\dfrac{d\theta(x,y)}{dy}\right\}$,

$$\lambda_1(x, y) = \frac{\lambda(x, y) - \lambda(\alpha, y)}{x - \alpha}; \quad \phi(x) = \phi_1(x)\,\phi_2(x),$$

$$F(x, y) = \theta(x, y)^2\,\phi_1(x) - \theta_1(x, y)^2\,\phi_2(x).$$

By (7), $\Sigma \int \dfrac{\lambda_1(y_n \cdot y)}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy$ = coefficient of $\dfrac{1}{x}$ in $\int \dfrac{\lambda_1(x, y)}{F(x, y)}\,dy$

$$= \text{coefficient of } \frac{1}{x} \text{ in } \int \frac{\lambda(x, y)}{(x - \alpha)\,F(x, y)}\,dy.$$

Because the lowest negative power of x in $\dfrac{1}{(x-\alpha)\,\mathrm{F}(x,y)}$ is $(n+1)$,

$$= \text{coefficient of } \frac{1}{x} \text{ in } \frac{2f(x)}{x-\alpha}\int \frac{\theta(x,y)\dfrac{d\theta_1(x,y)}{dy}-\theta_1(x,y)\dfrac{d\theta(x,y)}{dy}}{\theta(x,y)^2\phi_1(x)-\theta_1(x,y)^2\phi_2(x)}\,dy$$

$$= \text{coef. of } \frac{1}{x} \text{ in } \frac{fx}{(x-\alpha)\,[\phi(x)]^{\frac{1}{2}}} \log \frac{\theta(x,y)\,[\phi_1(x)]^{\frac{1}{2}}+\theta_1(x,y)\,[\phi_2(x)]^{\frac{1}{2}}}{\theta(x,y)\,[\phi_1(x)]^{\frac{1}{2}}-\theta_1(x,y)\,[\phi_2(x)]^{\frac{1}{2}}},$$

$$\int \frac{\lambda(\alpha,y)}{\mathrm{F}(\alpha,y)}\,dy = \int \frac{2f(\alpha)\left\{\theta(\alpha,y)\dfrac{d\theta_1(\alpha,y)}{dy}-\theta_1(\alpha,y)\dfrac{d\theta(\alpha,y)}{dy}\right\}}{\theta(\alpha,y)^2\phi_1(\alpha)-\theta_1(\alpha,y)^2\phi_2(\alpha)}\,dy$$

$$= \frac{f(\alpha)}{[\phi(\alpha)]^{\frac{1}{2}}} \log \frac{\theta(\alpha,y)\,[\phi_1(\alpha)]^{\frac{1}{2}}+\theta_1(\alpha,y)\,[\phi_2(\alpha)]^{\frac{1}{2}}}{\theta(\alpha,y)\,[\phi_1(\alpha)]^{\frac{1}{2}}-\theta_1(\alpha,y)\,[\phi_2(\alpha)]^{\frac{1}{2}}},$$

which agree with ABEL's results.

6682. (By H. G. DAWSON.)—If $(n-1)$ of the roots of $(abc\ldots)(x,1)^n$ are connected by the relation $\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=0$, show that the remaining root w is given by $z=aw+b$,

$$z^3-3\frac{(n-1)^2}{n+1}\cdot \mathrm{H}z+\frac{(n-2)(n-1)^3}{2(n+1)}\cdot \mathrm{G}=0,$$

where $\mathrm{H}=b^2-ac$, $\mathrm{G}=2b^3-3abc+a^2$; and hence deduce Quest. 6638.

Solution by Rev. T. R. TERRY, M.A.; CHRISTINE LADD; *and others.*

If $ax+b=y^n$, we have

$$y^n-\tfrac{1}{2}n(n-1)\mathrm{H}y^{n-2}+\tfrac{1}{3}n(n-1)(n-2)\mathrm{G}y^{n-3}+\ldots=0 \quad\ldots\ldots\ldots(1),$$

where $n-1$ roots are connected by the relation

$$\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=0,$$

which is equivalent to

$$(n-2)(n-3)\Sigma\alpha^3-3(n-3)\Sigma\alpha^2\beta+12\Sigma\alpha\beta\gamma=0\quad\ldots\ldots\ldots\ldots\ldots(2).$$

But if z be the remaining root of (1), we have obviously

$$\Sigma\alpha=-z,\quad \Sigma\alpha\beta=z^2-\tfrac{1}{2}n(n-1)\mathrm{H},$$

$$\Sigma\alpha\beta\gamma=-z^3+\tfrac{1}{2}n(n-1)\mathrm{H}z-\tfrac{1}{6}n(n-1)(n-2)\mathrm{G};$$

whence $$\Sigma\alpha^2\beta=2z^3-n(n-1)\mathrm{H}z+\tfrac{1}{2}n(n-1)(n-2)\mathrm{G},$$

and $$\Sigma\alpha^3=-z^3-\tfrac{1}{2}n(n-1)(n-2)\mathrm{G};$$

therefore, substituting in (2),

$$z^3-3\frac{(n-1)^2}{n+1}\cdot \mathrm{H}z+\frac{(n-2)(n-1)^3}{2(n+1)}\cdot \mathrm{G}=0.$$

6650. (By D. EDWARDES.)—If tangents are drawn to the circumscribed circle of a triangle ABC at the angular points, and if Δ be the area of the triangle so formed, and Δ_1 that of the orthocentric triangle of ABC, prove, with the usual notation, that $\Delta^{\frac{1}{2}} : \Delta_1^{\frac{1}{2}} = R : \rho$.

Solution by G. EASTWOOD, M.A.; J. O'REGAN; *and others.*

It is well known that the angles of the orthocentric triangle are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$, and it is obvious from a figure that the angles of the triangle formed by the tangents at A, B, C are equal to the same three quantities. Therefore these two triangles are similar, and therefore their areas are in the duplicate ratio of their inscribed circles.

6611. (By the Rev. T. R. TERRY, M.A. Suggested by Question 6590.)—If there be any distribution of mass M in space, and I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, A and B, respectively: prove that the moment of inertia about a parallel line through the centre of gravity of the mass is

$$\frac{a^2 I_a^2 + \ldots - (a^2 + b^2 - c^2) I_a I_b - \ldots - a^2 (b^2 + c^2 - a^2) M I_a - \ldots + M^2 a^2 b^2 c^2}{M (a^4 + b^4 + c^4 - 2a^2 b^2 - 2b^2 c^2 - 2c^2 a^2)}.$$

Solution by G. F. WALKER, M.A.; A. ANDERSON, B.A.; *and others.*

Let M be the mass, G the centre of gravity, and K the moment of inertia about it; $I_a = K + M \cdot AP^2$, $I_b = K + M \cdot BP^2$,

$$I_c = K + M \cdot CP^2.$$

Since A, B, C, G are four points in a plane, we have the following relation between their distances (d, e, f denote AG, BG, CG, respectively)—

$$a^2 (d^2 - e^2)(d^2 - f^2) + b^2 (e^2 - f^2)(e^2 - d^2) + c^2 (f^2 - d^2)(f^2 - e^2) + a^2 d^2 (a^2 - b^2 - c^2) + b^2 e^2 (b^2 - c^2 - a^2) + c^2 f^2 (c^2 - a^2 - b^2) + a^2 b^2 c = 0$$

(SALMON, p. 34). Whence (multiplying by M^2 and substituting),

$$a^2 (I_a - I_b)(I_a - I_c) + \ldots - a^2 (b^2 + c^2 - a^2) M (I_a - K) - \ldots + M^2 a^2 b^2 c^2 = 0,$$

$$\text{or } K = \frac{a^2 I_a^2 + \ldots - (b^2 + c^2 - a^2) I_b I_c - \ldots - a^2 (b^2 + c^2 - a^2) M I_a - \ldots + M^2 a^2 b^2 c^2}{M (a^4 + \ldots - 2b^2 c^2 - \ldots)}.$$

6201. (By CHRISTINE LADD.)—AB is the vertical diameter of a circle; ball descends down the chord AC, and, being reflected by the plane BC, escribes its path as a projectile; find the average range of the ball on e diameter CD, supposing all coefficients of friction relative to the descent of the ball on the chord to exist for which motion is possible.

Solution by W. J. C. SHARP, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

If AC make an angle α with AB, μ be the coefficient of friction, and r the radius of circle; we have

$$v \text{ (the velocity of impact)} = 2\left\{rg\cos\alpha(\cos\alpha-\mu\sin\alpha\right\}^{\frac{1}{2}},$$

and the velocity of reflection $= ev$, where e is the elasticity; the two equations $0 = ev\,.\,t - \frac{1}{2}g\sin 2\alpha\,.\,t^2$ and $s = \frac{1}{2}g\cos 2\alpha t^2$ determine the range s.

Therefore $$s = \frac{2e^2v^2\cos 2\alpha}{g\sin^2 2\alpha} = 8r\,\frac{e^2\cos 2\alpha\cos\alpha}{\sin^2 2\alpha}\left\{\cos\alpha-\mu\sin\alpha\right\}$$

therefore the average from $\mu=0$ to $\mu=1$

$$= \int_0^1 s\,d\mu = gr\,\frac{e^2\cos 2\alpha\cos\alpha}{\sin^2 2\alpha}\left\{\cos\alpha-\tfrac{1}{2}\sin\alpha\right\}$$
$$= e^2r\,(\cot^2\alpha-1)\,(2-\tan\alpha).$$

[It is assumed that both α and e are constants; if these are to be considered as variable, the results may be obtained from those given above by further integration.]

6492. (By J. HAMMOND, M.A.)—Prove that the sum of the infinite series $$\frac{1}{m}+\frac{1}{m(m+1)}+\frac{1}{m(m+1)(m+2)}+\ldots=\int_0^1 e^x(1-x)^{m-1}\,dx.$$

Solution by W. J. C. SHARP, M.A.; A. MCMURCHY, B.A.; *and others.*

$$\frac{x^m}{m}+\frac{x^{m+1}}{m(m+1)}+\frac{x^{m+2}}{m(m+1)(m+2)}+\&c.$$
$$=\left\{\left(\frac{d}{dx}\right)^{-1}+\left(\frac{d}{dx}\right)^{-2}+\left(\frac{d}{dx}\right)^{-3}+\&c.\right\}x^{m-1}$$
$$=\left(\frac{d}{dx}-1\right)^{-1}x^{m-1}=e^x\int_0^1 e^{-x}x^{m-1}\,dx;$$

therefore $$\frac{1}{m}+\frac{1}{m(m+1)}+\frac{1}{m(m+1)(m+2)}+\&c.$$
$$=e\int_0^1 e^{-x}x^{m-1}\,dx=\int_0^1 e^y(1-y)^{m-1}\,dy \quad \text{if } y=1-x.$$

6287. (By the EDITOR.)—The base AB of a triangle ABC being given in position and magnitude, and the side AC in magnitude only; trace the locus of the centre of the inscribed circle (1) generally, and (2) when AB=AC; also (3) find when the inscribed circle is a maximum, showing that then its radius is twice the distance of the third side of the triangle from the centre of the circumscribed circle; and state (4) what are the loci of the *escribed* circles.

I. *Solution by* W. J. C. SHARP, M.A.

1. Take the given side AB as axis of x, and A as origin of rectangular coordinates, and put $AC = b$, $AB = c$, and (x, y) for the centre of the inscribed circle; then we have

$$\tan \tfrac{1}{2}A = \frac{y}{x}, \quad \tan \tfrac{1}{2}B = \frac{y}{c-x}, \quad \tan \tfrac{1}{2}C = \frac{y}{b-x},$$

and $\tan \frac{1}{2}A \tan \frac{1}{2}B + \tan \frac{1}{2}B \tan \frac{1}{2}C + \tan \frac{1}{2}C \tan \frac{1}{2}A = 1$;

hence $y^2(b+c-x) = x(b-x)(c-x)$ or $(x^2+y^2)(b+c-x) = bcx$ .....(1)

is the equation to the locus, which is therefore a circular cubic through the origin, having the line $x = b+c$ for its asymptote.

2. When $b = c$ the equation becomes

$$(x^2+y^2)(x-2c) + c^2x = 0 \text{ or } x(x-c)^2 + y^2(x-2c) = 0 \text{ .....(2)},$$

which has a double point at $(c, 0)$, the tangents whereat are

$$c(x-c)^2 - cy^2 = 0, \text{ that is, } x+y-c = 0 \text{ and } x-y-c = 0.$$

Equation (2) may be written $(x^2+y^2)(x-c)^2 = c^2y^2$ ....(3);

hence the curve is such that, if P be any point on it,

$$AB \tan PBX = AP.$$

3. If the circle be a maximum, so is y^2, and the values of x are given by

$$2x^3 - 4(b+c)x^2 + 2(b+c)^2x - bc(b+c) = 0 \text{ .....(4)},$$

and if $b = c$ by $(x-c)(x^2-3cx+c^2) = 0$ .....(5).

The root $x = c$, of (5), gives y a minimum; and the other roots $x = \frac{1}{2}(3 \pm \sqrt{5})c$ give $y^2 = \dfrac{7 \pm 3\sqrt{5}}{1 \mp \sqrt{5}} c^2 = \frac{1}{2}(\mp 5\sqrt{5} - 11)c^2$, so that $\frac{1}{2}(3+\sqrt{5})c$ makes y impossible, and $x = \frac{1}{2}(3-\sqrt{5})c$ makes y a maximum.

4. For the circle touching AB and the other sides produced, we have

$$\tan \tfrac{1}{2}A = -\frac{x}{y}, \quad \tan \tfrac{1}{2}B = -\frac{c-x}{y}, \quad \tan \tfrac{1}{2}C = -\frac{y}{b+x};$$

and the equation is $(x^2+y^2)(x+b-c) = bcx$.....(6),

which is also a circular cubic through the origin, whose real asymptote is $x+b-c = 0$, the satellite of infinity being, as before, $x = 0$.

If $b = c$, the curve degenerates into a circle whose centre is A and radius c, and the axis of y.

The locus of the circle touching AC and the other sides produced is the similar curve $(x^2+y^2)(x+c-b) = bcx$ .....(7),

which becomes identical with (6) if $b = c$.

The equation to the locus of the centre of the circle escribed to the side BC is $(x^2+y^2)\{x-b-c\} + bcx = 0$.....(8),

which is the same curve as the locus (1) of the centre of inscribed circle.

5. It is evident that, in each of the loci, the origin is the double focus; that, in (2), $(c, 0)$ is a focus; that, in (1), (6), (7), the tangents at the points where $y = 0$ cuts the curves are, respectively,

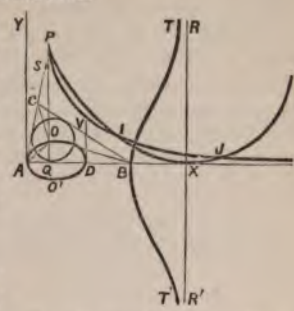
$$(x = 0,\ x-b = 0,\ x-c+0), \quad (x = 0,\ x+b = 0,\ x-c = 0),$$

$$(x = 0,\ -b = 0,\ x+c = 0);$$

and that the corresponding satellites in (1), (6), (7) are $x-b-c = 0$, $x+b-c = 0$, $x-b+c = 0$, that is to say, in each case, the asymptote.

II. *Solution by the* PROPOSER.

In the triangle ABC, let O be the centre of the inscribed circle, and OQ ($=y$) its radius; then, if $AC = AD = b$, $AB = c$, $BC = z$, $AX = b + c = e$, $AO = r$, $\angle OAB = \theta$, and $AQ = x$, so that (x, y) and (r, θ) are the rectangular and polar coordinates of O, we have



$$\tfrac{1}{2}(e-z) = x = y\cot\theta \quad \ldots\ldots(1),$$

$$\tfrac{1}{2}(e+z) = \frac{bc}{2y}\sin 2\theta = \frac{bc}{x}\cos^2\theta \ldots(2);$$

whence, eliminating z, θ, we find, for the locus of O, any one of the following equations :—

$$(e-x)\,r^2 = bcx, \qquad (e-x)\,y^2 = x\,(x-b)\,(x-c) \quad \ldots\ldots\ldots\ldots(3, 4),$$

$$(r^2+bc)\cos\theta = er, \qquad \frac{dy}{dx} = \frac{\frac{1}{2}bce - x\,(e-x)^2}{y\,(e-x)^2} \quad \ldots\ldots\ldots\ldots(5, 6).$$

This cubic locus consists of a loop AODO′, and (in general) two infinite branches TBT′ along the perpendicular asymptote RXR′. When $b = c$, the curve is continuous at B, where the two branches cross each other. Differentiating, in (2), with respect to θ, we find

$$\frac{2bc}{y}\cos 2\theta - \frac{bc}{y^2}\sin 2\theta\,\frac{dy}{d\theta} = \frac{dz}{d\theta} = \frac{2bc}{z}\sin 2\theta,$$

the second equality resulting from differentiating

$$(1)\times(2) \equiv c^2 - z^2 = 2bc\,(1+\cos 2\theta);$$

whence we obtain

$$\frac{dy}{d\theta} = \frac{2y}{z}(z\cot 2\theta - y) \quad \ldots\ldots\ldots\ldots(7);$$

hence, when the inscribed circle is a maximum, its radius

$$y = z\cot 2\theta \quad \ldots\ldots\ldots\ldots(8),$$

which proves part (3) of the Question.

From (1), (2), (8), we have $x = (e-x)\cos 2\theta$, $2x\,(e-x)^2 = bce$...(9, 10). Equation (10) determines the maximum radius of the inscribed circle, agreeing with Mr. SHARP's equation (4), and with what, in another way, is obtainable from (6) above. From (10) we have, in the figure,

$$AB\,.\,AD\,.\,AX = 2AQ\,.\,QX^2 \quad \ldots\ldots\ldots\ldots(11).$$

Hence, to determine the triangle that has the maximum inscribed circle, we have the following construction :—

With AX as parameter and XR as axis draw the parabola PIXJ; and through the point V (where DV is perpendicular to AX and $= \frac{1}{2}AB$) along the asymptotes AX, AY, draw the rectangular hyperbola PVIJ; from P draw POQ perpendicular to AX; and construct the triangle ABC so as to have the two given sides AB, AC, and the third side $BC = AX - 2AQ$; then the circle inscribed in this triangle will be a maximum.

For, from the parabola and the hyperbola respectively, we have

$$AX\,.\,PQ = QX^2, \quad AD\,.\,DV = AQ\,.\,QP \text{ or } AD\,.\,AB = 2AQ\,.\,PQ;$$

hence $AB\,.\,AD\,.\,AX = 2AQ\,.\,QX^2$, which agrees with (11).

After determining P, and Q therefrom, as above, the triangle may be

otherwise constructed, by the aid of equation (9), by inflecting from A to PQ a line AS = QX, taking thereon AC = b, and joining BC.

Writing (10) in the form $F(x) \equiv 2x(e-x)^2 - bce = 0$, we find that, when $x = 0, b, c, e, \frac{4}{3}e$, then $F(x) = -bce$, $(c-b)bc$, $-(e-b)bc$, $-bce$, $\frac{1}{27}e\{8(e-b)^2+5bc\}$; hence (since values substituted for x in an equation $F(x) = 0$ give *like* signs when they comprise between them an *even* number of roots, and *unlike* signs when they comprise an *odd* number) the roots of the cubic (10), that is, of $F(x) = 0$, are in the intervals $(0, b)$, (b, c), $(e, \frac{4}{3}e)$, therein agreeing with what is shown in the above graphical construction.

From (8) we have $y \tan A = z = y(\cot \frac{1}{2}B + \cot \frac{1}{2}C)$; hence, in the triangle that has the maximum inscribed circle, there are the following relations among the angles:—

$$\tan A = \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C, \quad \cos A = 2 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

$$1 + \cos A = \cos B + \cos C, \quad \cos B + \cos C - \cos A = 1.$$

The infinite branch TBT′ of the cubic locus is the locus of the circle escribed to the side BC, and the locus of the inscribed circle is the loop AODO′.

6745. (By A. McIntosh, B.A.)—If in a nodal cubic, with the nodal tangents at right angles, a right-angled triangle be inscribed having the right angle at the double point; show that (1) the hypotenuse passes through a fixed point on the curve; and (2) in the case of the Folium of Descartes, this point is the vertex or apse of the loop.

Solution by Professor Townsend, F.R.S.

The equation of the curve, to the nodal tangents as axes of coordinates, being $u_3 + xy = 0$, where $u_3 = ax^3 + by^3 + 3hx^2y + 3ky^2x$, that of the three lines from the origin to its three intersections with any arbitrary line $lx + my = 1$ is consequently $u_3 + (lx + my)xy = 0$, which, when two of them are (as by hypothesis) at right angles to each other, must be of the form $(px + qy)(x^2 - y^2 + 2rxy) = 0$; hence we must have $p = a$ and $q = -b$, which shows that the third line is fixed, and therefore &c., as regards the general property in question, from which the particular follows immediately from the consideration that, where the curve has an axis of figure, the point must be similarly related to its two symmetrical halves, and be therefore its apsidal point on the axis.

A nodal cubic reciprocating, to any circle having its centre at the node, into a tricuspidal quartic having a double tangent at infinity; and inverting, to the same, into a unicircular quartic having a triple point at the node; we infer consequently from the above, by application of the latter and former processes respectively, that—

(*a*) A quartic, having a triple point, being supposed to intersect infinity at the two circular points, and also at a real pair of rectangular points; the circle passing through the triple point, and through the extremities of a variable chord subtending it at right angles, passes through a second fixed point on the curve.

(*b*) A quartic, having a triad of cuspidal points, being supposed to have double contact with infinity at a pair of rectangular points; every tangent to either branch into which it is divided by its bitangent chord intersects with the orthogonal tangent to the opposite branch on a fixed tangent to the curve.

Every right angle, whatever be its position, being divided harmonically by the connectors of its vertex with the two circular points at infinity in its plane; we infer again, consequently, from the above, by projection in its most general sense, that—

(*c*) A variable chord of a nodal cubic, divided harmonically in every position by any fixed pair of nodal chords harmonically conjugate to each other with respect to the nodal tangents, passes in every position through a fixed point on the curve; that, viz., at which (see 4995) the tangents at the opposite extremities of the two nodal chords intersect on the curve.

6373. (By Professor SYLVESTER, F.R.S., D.C.L.)—If

$$fx = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} \ldots + L$$

$$= ax^n + nbx^{n-1} + n \cdot \tfrac{1}{2}(n-1)\, cx^{n-2} + \frac{n(n-1)(n-2)}{1.2.3} dx^{n-3} \ldots + l,$$

prove that (1) fx cannot have more real roots than there are continuations of sign in the series $A^2 : B^2 - AC : C^2 - BD : \ldots : L^2$; (2) fx cannot have more real roots than there are continuations of sign in the series

$$b^2 : c^2 - \tfrac{3}{4}bd : d^2 - \tfrac{2}{3}ce : e^2 - \tfrac{3}{2}\tfrac{4}{5}df : f^2 - \tfrac{3}{1}\tfrac{5}{6}eg : \ldots : l^2.$$

Solution by the PROPOSER.

The above conclusions follow from putting, in the formula $\gamma_r = \dfrac{C-1+r}{C+r}$, in TODHUNTER'S chapter "On NEWTON'S Rule and SYLVESTER'S Theorem" in his "Theory of Equations" (p. 249), for (1) $C = \infty$, for (2) $C = 0$.

Observe that C is limited to be *anything* not *between* 0 and $-n$, or, as we may phrase it, anywhere on the arc of quantities from 0 to $-n$, passing through $\pm\infty$.

NEWTON'S Rule corresponds to putting C equal to *one* extremity of this circle, viz. $-n$. Theorem (1) corresponds to putting C = *to the half-way quantity*, viz. $\pm\infty$. Theorem (2) corresponds to putting C = to the *other* extremity, viz. 0.

As an example of (1), take $x^n + x^{n-1} + \ldots + n + 1 = 0$. NEWTON'S rule only indicates the existence of some imaginary roots; then, considering 0 as $\pm$ at will, (1) proves that all the roots, or all but one, are imaginary.

Nobody seems to have noticed that, when we take $C = -n$ (NEWTON'S rule case), the quadratic quantities $G_1, G_2, \ldots G_{n-1}$ intermediate between the first and last, which are merely the pillar-post signs $+\ +$, instead of being of the dimensions

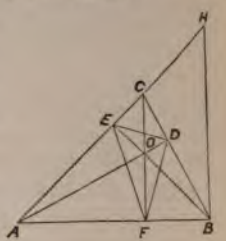
$$2(n-1),\ 2(n-2),\ \ldots,\ 2.2,\ 2.1 \text{ (each)},$$

lose two dimensions, becoming of the respective degrees

G_{n-1}, G_{n-2}, G_2, G_1, $2(n-2)$, $2(n-3)$, ..., 2, 0, G_1 being a constant.

5546. (By the Editor.)—In a triangle ABH, right-angled at B and having AB = BH, take HC = ⅓HA, and join BC; and let O be the orthocentre of the triangle ABC, DEF its orthocentric triangle, and a', b', c', ρ, Δ' the sides and inscribed radius of this orthocentric triangle, the usual notation referring to the parts of the triangle ABC; then prove that (1) tan A = 1, tan B = 2, tan C = 3; (2) $\Delta = 5\Delta' = b^2 - a^2 = \frac{1}{2}c^2$; (3) AF = 2FB, BD = ⅔DC, CE = ⅓EA; (4) AO = 5OD, BO = 2OE, CO = OF; (5) $\rho = \frac{1}{5}$R; and (6) develop other properties of the system.

Solution by G. Heppel, M.A.; J. O'Regan; *and others.*



In Booth's *New Geometrical Methods* (Vol. II., p. 301) it is shown that

$$\frac{\Delta'}{\Delta} = \frac{\rho}{R} = 2\cos A \cos B \cos C \ldots\ldots\ldots(1).$$

We have also AD = CD tan C, OD = CD cot B,

therefore $$\frac{AD}{OD} = \tan B \tan C;$$

or AO = OD (tan B tan C − 1),
similarly BO = OE (tan C tan A − 1),
CO = OF (tan A tan B − 1), } .....(2).

And obviously AF : FB = tan B : tan A,

BD : DC = tan C : tan B, CE : EA = tan A : tan C.....(3).

Now, from the figure, AF = CF = 2BF, *i.e.*, tan A = 1, tan B = 2, and tan C = −tan (A + B) = 3.

Substituting for A, B, C in (1), (2), (3), we obtain

$\Delta = 5\Delta'$, $\rho = \frac{1}{5}$R, AO = 5 . OD, BO = 2 . OE, CO = OF, AF = 2FB,

BD = ⅔DC, CE = ⅓EA.

Again, $\Delta = \frac{CF}{HB}$. ABH = ⅔ABH = $\frac{1}{3}c^2$, and $b^2 - a^2 = AF^2 - BF^2 = \frac{1}{3}c^2$.

6571 & 6598. (By C. B. S. Cavallin, M.A.)—Taking the variation of the Croftonian integral $\iint (\theta - \sin\theta)\, dx\, dy = \frac{1}{2}L^2 - \pi\Omega$, (*Phil. Trans.*, 1868, p. 188), on the supposition that the contour of reference (of length L inclosing an area Ω) changes to another nearly situated, generated by prolonging each radius ρ of curvature in Ω a length $\mu f(\rho)$, where μ is an infinitely small constant, prove that we obtain

$$\iint \left\{ \frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right\} \sin^2 \tfrac{1}{2}\theta \, dx\, dy = \tfrac{1}{2}L \int^{2\pi} f(\rho)\, d\phi,$$

where t_1, t_2 are the lengths of the tangents, drawn from the point (x, y) to Ω; ρ_1, ρ_2 the radii of curvature at the corresponding points of contact; ρ the radius of curvature at an arbitrary point of the curve; and ϕ the inclination of this radius to a fixed line in the plane; the integration extending over the whole plane outside Ω.

(6598.) Prove that, (1) by starting from the Croftonian integral

$$\iint \theta\, dx\, dy = \pi\Theta - \int_0^{2\pi} \Sigma\, d\omega, \quad (i\delta., \text{p. } 190)$$

and only varying the *inner* contour, we get

$$\iint \left\{ \frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right\} dx\, dy = \int_0^{2\pi} cf(\rho)\, d\omega,$$

where θ is the angle which at any point within the annulus (Θ) between an interior and exterior contour is subtended by the former; c is a chord of the latter intercepting a segment Σ of it and touching the former; ω the angle which c makes with a fixed line, and the other notation as in Quest. 6571. Also, (2) by only varying the *outer* contour, prove that we get

$$\int_0^{2\pi} (\pi - \theta) \rho f(\rho)\, d\phi = \int_0^{2\pi} \left\{ \int_0^{\beta} \rho f(\rho)\, d\phi \right\} d\omega,$$

where ρ is now the radius of curvature of the outer contour at the point at which the inner subtends an angle θ; β the angle between the normals at the ends of c, and the rest of the notation as in (6571), but the integration extends only over the annulus between the two contours.

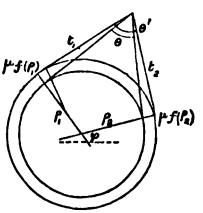
Solution by J. J. Walker, M.A.

Calling the area of the new contour Ω', since in the space between the two contours $\theta - \sin\theta$ has the constant value π, the sinister of the given equality must be increased by the term $\pi(\Omega' - \Omega)$ if the integration be extended only over the space exterior to the new contour. The variation thus gives (θ', L′ referring to the new contour)

$$\iint (\theta' - \theta - \sin\theta' + \sin\theta)\, dx\, dy = \mathrm{L}(\mathrm{L}' - \mathrm{L}),$$

But the figure shows at once that

$$\theta' - \theta = \mu \left(\frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right),$$

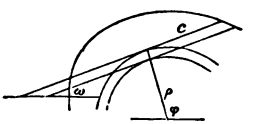


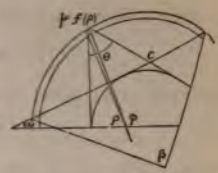
and $\sin\theta' - \sin\theta = (\theta' - \theta)\cos\theta$. Again, if the elements of L′ and L limited by two consecutive normals, inclined at an angle $d\phi$, be considered, their difference is plainly equal to $\mu f(\rho)\, d\phi$, so that

$$\mathrm{L}' - \mathrm{L} = \mu \int_0^{2\pi} f(\rho)\, d\phi.$$

Substituting these values in the equality above, and dividing by 2μ, the first result is proved.

For the second result, the sinister of the equality from which we are to start must be increased by the term $\pi(\Theta - \Theta')$, if the integration is to extend only over the annulus between the new interior contour and the exterior contour, since within the two interior contours θ has the constant value π; so that the variation gives $\iint (\theta' - \theta)\, dx\, dy = \int_0^{2\pi} (\Sigma - \Sigma')\, d\omega$. But, as above, $\theta' - \theta = \mu(\ldots)$ and $\Sigma - \Sigma'$ plainly is equal to $\mu cf(\rho)$. Substituting and dividing by μ, the second result is obtained.





For the third result, the variation of the sinister in this case—since at corresponding points on the new and given exterior contours θ may be regarded as having the same values, retaining only terms multiplied by the first power of μ—will be $\iint \theta\, dx\, dy$, taken only over the space between those contours, which is plainly equal to $\mu \int_0^{2\pi} \theta \rho f(\rho)\, d\phi$. For the variation of the dexter $\pi(\Theta' - \Theta) = \pi$ times the same integral; and $\Sigma' - \Sigma$ (which is the portion cut off by c of the elementary annulus contained between the new and given exterior contours) is similarly equal to $\mu \int_0^{\beta} \rho f(\rho)\, d\phi$. Substituting these values, transposing the term multiplied by π, and dividing by $-\mu$, the final result, as given in Quest. 6598, is obtained.

[Mr. CAVALLIN is to be congratulated on having obtained these new integral relations in so ingenious a manner from Prof. CROFTON's integrals.]

6651. (By E. W. SYMONS, B.A.)—A parabola has contact of the 4th order with a given circle; prove that its focus moves on one concentric circle, and that its directrix touches another.

Solution by the Rev. T. R. TERRY, M.A., F.R.A.S.; E. RUTTER; *and others.*

Since the contact is of the 4th order, the point of contact must be at the vertex of the parabola. But radius of curvature at the point $= 2SA$. Hence, if R be the radius of the given circle, the locus of the focus is a concentric circle of radius $\frac{1}{2}$R, and envelope of directrix is a concentric circle of radius $\frac{3}{2}$R.

6673. (By the Rev. W. A. WHITWORTH, M.A.)—Find a triangle whose altitude and sides are expressed by four consecutive integers, and show that only one such triangle exists.

Solution by the PROPOSER.

If we first tabulate all the triangles whose sides are expressed by consecutive integers and whose areas are rational, we find

3, 4, 5 (area = 6); 13, 14, 15 (area = 84); 51, 52, 53 (area = 1170); &c.

The second of these (13, 14, 15) has an altitude 12, and thus fulfils the conditions of the question. And it is plain that no triangle whose sides are *higher* consecutive integers can possibly fulfil the conditions. For, as we proceed to higher integers, such series of triangles will approximate

more and more to an *equilateral* triangle, and the ratio of the perpendicular to any of the sides must approximate to the ratio $\sqrt{3} : 2$, or approximately 6 : 7. But the ratio of an integer to any one of the next three consecutive integers recedes from this value when the integer is higher than 18. Therefore it is impossible that the conditions of the question can be fulfilled by any higher numbers. Hence the only triangle is the triangle whose altitude is 12, and its sides 13, 14, 15.

6427. (By R. A. Roberts, M.A.)—Show that the cubics whose equations in rectangular coordinates are

$$(a-b)\,xy^2+g\,(y^2-x^2)-2fxy-k^2\,(ax+g)=0,$$
$$(a-b)\,x^2y+f\,(y^2-x^2)+2gxy+k^2\,(by+f)=0,$$

cut each other at right angles at their seven finite points of intersection.

Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others.*

We have, from the cubics,

$$[2xy\,(a-b)+2gy-2fx]\,\frac{dy}{dx}+(a-b)\,y^2-2gx-2fy-k^2a=0,$$

and $[(a-b)\,x^2+2fy+2gx+k^2b]\,\frac{dy}{dx}+2xy\,(a-b)+2gy-2fx=0;$

and, if they are at right angles, we must have $xy\,(a-b)+gy-fx=0$, or $(a-b)x^2+2fy+2gx+k^2b+(a-b)y^2-2gx-2fy-k^2a=0$, *i.e.*, $(x^2+y^2)=k^2$.

Now, if we multiply the first equation by y, and the second by x, and add, we get $(x^2+y^2)\,xy\,(a-b)+(gy+fx)\,(y^2-x^2)$
$+2xy\,(gx-fy)-k^2xy\,(a-b)-k^2\,(gy-fx)=0,$

which is $(x^2+y^2-k^2)[xy\,(a-b)+gy-fx]=0;$

and therefore the curves cut at right angles.

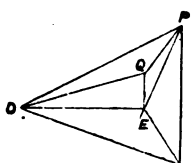
6570. (By Prof. Genese, M.A.)—The angle which a fixed diameter of a rectangular hyperbola subtends at a variable point is divided into two parts, whose sines are in a given ratio; prove that the dividing line passes through a fixed point.

Solution by A. McMurchy, B.A.; Christine Ladd; *and others.*

Let the dividing line meet the curve again at Q, and let DE be the fixed diameter, then

$$\angle QDP = QEP$$

or supplement; hence diameter of circle PQD
= PQ cosec QDP = diameter of circle QED;
$\therefore$ QD : QE = sin QPD : sin QPE = constant,
and Q is a fixed point. There will be another fixed point corresponding to the lines dividing the angle DPF externally.



6725. (By W. H. H. Hudson, M.A.)—If Δ be the area, and P the perimeter, of a triangle drawn on a sphere of radius r, and if Δ', P' be the area and perimeter of the polar triangle; prove (1) that

$$\frac{\Delta}{r^2}+\frac{P'}{r}=\frac{\Delta'}{r^2}+\frac{P}{r}=2\pi;$$

and (2) state the corresponding proposition for any spherical polygon.

Solution by R. Tucker, M.A.; Christine Ladd; *and others.*

$$\frac{\Delta}{r^2}=A+B+C-\pi,\quad P'=a'+b'+c';$$

therefore $\frac{\Delta}{r^2}+P'=(A+a')+(B+b')+(C+c')-\pi-2\pi=\frac{\Delta'}{r^2}+P.$

(2) $$\frac{S}{r^2}=\Sigma-(n-2)\pi,\quad P'=a'+b'+c'+\ldots;$$

$$\therefore\ \frac{S}{r^2}+P'=(A+a')+(B+b')+\ldots-(n-2)\pi=n\pi-(n-2)\pi=2\pi=\frac{S}{r^2}+P.$$

[With a suitable definition of polar polygon, the same enunciation does for all cases.]

2706. (By J. Griffiths, M.A.)—Prove that (1) two real equilateral hyperbolas can be drawn to touch the sides of a given obtuse-angled triangle, and to pass through the centre of its circumscribed circle; and (2) the centres of these curves are the points of intersection of the nine-point and circumscribed circles of the triangle.

Solution by Prof. Wolstenholme, M.A.

When a rectangular hyperbola touches the sides of a given triangle, it is a well-known property that its centre lies on the polar circle of the triangle; also, when a conic is inscribed in a given triangle and passes through the centre of the circumscribed circle, it has been proved (in the Solution of Quest. 2726, on p. 60 of Vol. XI. of *Reprint*) that the director circle touches the circumscribed circle of the triangle; or, when the conic is a rectangular hyperbola, its centre lies on the circumscribed circle. Hence for a rectangular hyperbola touching the sides of a given triangle, and passing through the centre of the circumscribed circle, the centre will be the two common points of the circumscribed circle and polar circle, *i.e.*, of the circumscribed and nine-point circle, and these are real points (and therefore the hyperbolas real) when the triangle is obtuse-angled.

[It is interesting to notice how it happens there are only two solutions instead of four. The locus of the centre of a conic inscribed in a given triangle and passing through a given point, is a conic touching the straight lines joining the middle points of the sides of the given triangle; and this conic generally intersects the polar circle in four points; but when the given point is the centre of the circumscribed circle, these four coalesce

two and two, and the locus has double contact with the polar circle at points lying on the circumscribed and nine-point circles. The locus of the centre of a rectangular hyperbola touching the sides of the triangle of reference, is the polar circle $x^2 \cot A + y^2 \cot B + z^2 \cot C = 0$; and the locus of the centre of a conic inscribed in the triangle of reference and passing through the centre of the circumscribed circle, is the conic

$$\{(y+z-x)\sin 2A\}^{\frac{1}{2}} + \ldots + \ldots = 0, \text{ or}$$

$$x^2 \cot A + y^2 \cot B + z^2 \cot C - \sin A \sin B \sin C\,(x \cot A + y \cot B + z \cot C)^2 = 0,$$

a conic touching the polar circle at the two points where the straight line $x \cot A + y \cot B + z \cot C = 0$ meets it. But

$$x \cot A + y \cot B + z \cot C = 0$$

is the common radical axis of the circumscribed, nine-point, and polar circles of the triangle; the circumscribed and nine-point circles being

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = (x+y+z)\,(x \cot A + y \cot B + z \cot C),$$

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = \tfrac{1}{2}(x+y+z)\,(x \cot A + y \cot B + z \cot C).$$

The common points of these circles are real when $\cot A \cot B \cot C$ is negative; that is, when the triangle is obtuse-angled, or when the polar circle is real.]

6338. (By W. H. H. Hudson, M.A.) — From a semicircle, whose diameter is in the surface of a fluid, a circle is cut out, whose diameter is the vertical radius of the semicircle; find the centre of pressure of the remainder.

I. *Solution by* W. J. C. Sharp, M.A.; A. Anderson, B.A.; *and others.*

The centre of pressure is evidently in the vertical diameter, and the pressure on a strip of width dx at a depth x

$$\propto \{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\}\, x\, dx,$$

where r is the radius of the semicircle; hence we have

$$\bar{x} = \frac{\int_0^r \{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\}\, x^2\, dx}{\int_0^r \{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\}\, x\, dx} = \frac{\frac{\pi r^4}{16} - \frac{5\pi r^4}{128}}{\frac{r^3}{3} - \frac{\pi r^3}{16}} = \frac{9\pi r}{8(16-3\pi)}.$$

II. *Solution by the* Proposer.

The depth of the centre of pressure of a plane area immersed vertically in a fluid is $k^2 \div h$, where k is the radius of gyration of the area about the line of floatation, h the depth of the centre of gravity. For the semicircle (radius r) $k^2 = \frac{1}{4}r^2$, $h = \frac{4r}{3\pi}$; and for the circle (radius $\frac{1}{2}r$) $k^2 = \frac{5}{16}r^2$, $h = \frac{1}{2}r$,

also, pressure on semi-circle : pressure on circle

$$= \frac{\pi r^2}{2} \cdot \frac{4r}{3\pi} : \frac{\pi r^2}{4} \cdot \frac{r}{2} = 16 : 3\pi;$$

therefore depth of centre of pressure of remainder

$$= \frac{16 \cdot \frac{3}{8}\pi - 3\pi \cdot \frac{3}{8}\pi}{16 \cdot - 3\pi} r = \frac{9\pi r}{8(16-3\pi)}.$$

6423. (By Rev. T. W. Openshaw, M.A.) — If a pair of conjugate diameters CP, CD of an ellipse be produced to meet the auxiliary circle in Q, Q', and the ordinates QN, Q'N' meet the ellipse in R, R'; prove (1) that tangents at R, R' intersect on the director-circle: and (2) extend this also to the ellipsoid; also, if the ordinates of P and D meet the auxiliary circle at K, K', prove (3) that the tangents to the auxiliary circle at K, K' meet on the auxiliary circle of the ellipse on which the tangents at P and D intersect.

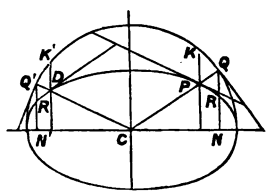
Solution by G. F. Walker, M.A.; N. Sarkar, B.A.; *and others.*

The relation $\tan\phi \tan\phi' = -\frac{b^2}{a^2}$ can be easily put in the form

$$a^2 \sin\phi \sin\phi' + b^2 \cos\phi \cos\phi' = 0,$$

$$a^2 [\cos^2 \tfrac{1}{2}(\phi - \phi) - \cos^2 \tfrac{1}{2}(\phi + \phi')]$$
$$+ b^2 [\cos^2 \tfrac{1}{2}(\phi - \phi') - \sin^2 \tfrac{1}{2}(\phi + \phi')] = 0;$$

giving $a^2 \left(1 - \frac{x^2}{a^2}\right) + b^2 \left(1 - \frac{y^2}{b^2}\right) = 0,$



or the auxiliary circle as the locus of the intersection of tangents at R and R'. The point of intersection of the tangents at K and K', and the point of intersection of the tangents at P and D will lie on a line perpendicular to the major axis, and the latter will divide the perpendicular from the former in the ratio $b : a$ (segments measured from the major axis). The locus of the latter point is known to be an ellipse similar and similarly situated to the given ellipse, and therefore the tangents at K and K' will meet in its auxiliary circle.

In the case of the ellipsoid, if l_1, m_1, n_1, &c. be the conjugate diameters, $\frac{al_1}{r_1}, \frac{bm_1}{r_1}$, &c. will be the other three points.

If (x, y, z) be the point of concourse of the tangents, we have

$$\frac{x}{a} \left| \frac{al_1}{r_1}, \frac{bm_1}{r_1}, \frac{cn_1}{r_1} \right| = \left| 1, \frac{bm_1}{r_1}, \frac{cn_1}{r_1} \right|,$$

and reducing by the known relations between the l, m, n's, we have

$$\frac{abcx}{ar_1r_2r_3} = bc \left(\frac{l_1}{r_2r_3} \ldots\right), \text{ and } x = l_1r_1 + l_2r_2 + l_3r_3;$$

hence $x^2 + y^2 + z^2 = (l_1r_1 + l_2r_2 + l_3r_3)^2 \ldots = r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$, and the locus is the director sphere.

6486. (By J. R. Wilson, M.A.)—From any point P on a central conic lines are drawn to the extremities of the axis BB′. BQ perpendicular to BP meets PB′ in Q, and B′R perpendicular to B′P meets PB in R. Show that the envelope of QR is a central conic, the ratio of whose axes is $a^2+b^2 : 2ab$.

Solution by R. Knowles, B.A., L.C.P.; W. J. C. Sharp, M.A.; *and others.*

Let the coordinates of P be $a\cos\phi$, $b\sin\phi$; then, forming the equations to PB, B′R; B′P, BQ, we find the coordinates of R and Q, and thence the equation to QR, which is

$$(a^4-b^4)\,y\cos\phi+2ab\,(a^2-b^2)\,x\sin\phi+2ab^2\,(a^2+b^2)=0,$$

whose envelop is $4a^2b^2\,(a^2-b^2)^2\,x^2+(a^4-b^4)^2\,y^2=4a^2b^4\,(a^2+b^2)^2$,

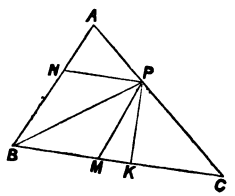
the semi-axes $b\,.\,\dfrac{a^2+b^2}{a^2-b^2}$ and $\dfrac{2ab^2}{a^2-b^2}$, and the areas as $a^2+b^2 : 2ab$.

6631. (By Prof. Matz, M.A.)—Show that the average area of parallelograms inscribed in a triangle is one-third of the area of the triangle.

Solution by C. Morgan, B.A.; Belle Easton; *and others.*

Let CP $= s$; then area of parallelogram NPMB

$$= \text{PK}\,.\,\text{BM} = s\,.\,\sin\text{C}\,.\,\frac{a}{b}\,(b-s);$$



hence the mean area is therefore

$$\frac{a}{b}\sin\text{C}\int_0^b s\,(b-s)\,ds \div b$$

$$= \frac{a}{b^2}\sin\text{C}\int_0^b \left[\tfrac{1}{2}b^3-\tfrac{1}{3}b^3\right]$$

$$= \tfrac{1}{6}ab\,.\,\sin\text{C} = \tfrac{1}{3}\text{ of the area of the triangle ABC.}$$

[See the Editor's Question 1217.]

6240. (By C. Leudesdorf, M.A.)—A homogeneous cube, whose edge is $2a$, strikes with one of its angular points against a perfectly rough inelastic wall. Just before impact the cube was moving in a given direction with velocity v, and rotating about an axis through its centre parallel to this direction with angular velocity ω. Prove that, if the centre of the cube begins, after the impact, to move with velocity v' in a direction making an angle θ with its direction before impact, then

$$\cos\theta = \frac{33vv'}{4a^2\omega^2+27v^2}.$$

Solution by G. F. Walker, M.A.; D. Edwardes; *and others.*

Let the point of impact be taken as origin, and the three edges which meet in that point as axes of coordinates. Let I be the moment of inertia of the cube about a line through its centre of gravity; α, β, γ the angles which the direction before impact makes with the axes, ω_x, ω_y, ω_z the angular velocities after impact about the axes. The velocity of the centre after impact along the axis of x will be therefore $a(\omega_y - \omega_z)$, and similarly for the axes of y and z. Hence

$$\cos\theta = av^{-1}\{\cos\alpha(\omega_y-\omega_z) + \cos\beta(\omega_z-\omega_x) + \cos\gamma(\omega_x-\omega_y)\},$$

and the dynamical equations will be, with two similar equations,

$$I\omega\cos\alpha + av(\cos\gamma - \cos\beta) = (I+2a^2)\omega_x - a^2(\omega_y+\omega_z);$$

hence $$\cos\theta = \frac{a^2v(2-2\cos\alpha\cos\beta-2\cos\beta\cos\gamma-2\cos\gamma\cos\alpha)}{(I+3a^2)v'}.$$

Also, subtracting the equations, squaring, and adding, we get

$$(I+3a^2)^2\frac{v'^2}{a^2} = (2-2\cos\alpha\cos\beta-2\cos\beta\cos\gamma-2\cos\gamma\cos\alpha)(I^2\omega^2+3a^2v^2);$$

therefore $$\cos\theta = \frac{(I+3a^2)vv'}{I^2\omega^2+3a^2v^2};$$ and, putting $I = \frac{2}{3}a^2$, we have &c.

6029. (By W. J. C. Sharp, M.A.)—Prove that the covariant

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta$$

is the locus of points whose polar conics are parabolas, and separates the points whose polar conics are ellipses from those whose polar conics are hyperbolas.

Solution by the Proposer.

If $ax+\beta y+\gamma z = 0$ be the equation to the line at infinity, the conic $ax^2+by^2+cz^2+2fyz+2gxz+2hxy = 0$ will meet it in two real, identical, or imaginary points, according as $A\alpha^2+B\beta^2+$ &c. is negative, zero, or positive; that is, the polar conic is an hyperbola, a parabola, or an ellipse according to the sign of $A\alpha^2$, &c., which proves the proposition.

6453. (By J. L. McKenzie, B.A.)—Three consecutive numbers may readily be found, each of which contains a square factor > 1; *e.g.*, 48, 49, 50; 98, 99, 100; 124, 125, 126; &c. Is the same possible for four consecutive numbers? If so, find the first case in which it occurs.

Solution by G. Heppel, M.A.; C. Bickerdike; *and others.*

It is possible, and the next set of three occurring after those mentioned forms part of a sequence of four, the numbers being 242, 243, 244, 245.

6272. (By E. ANTHONY, M.A.)—The normal at any point P of a curve intersects two straight lines, which meet in the point O, and which are at right angles to one another, in the points G and g, so that we have

$$m \cdot OG^2 \cdot Pg = n \cdot Og^2 \cdot PG;$$ find the equation to the curve.

Solution by C. MORGAN, B.A; J. YOUNG, B.A.; *and others.*

Taking the straight lines as axes of coordinates, the intercepts on them by the normal are $\frac{x\,dx + y\,dy}{dx}$, $\frac{x\,dx + y\,dy}{dy}$; therefore $\frac{Og^2}{OG^2} = \left(\frac{dx}{dy}\right)^2$. And $PG = y\frac{ds}{dx}$, $Pg = x\frac{ds}{dy}$; therefore $\frac{Pg}{PG} = \frac{x}{y}\frac{dx}{dy}$. Hence, by the given relation, $\frac{dx}{dy} = 0$, whence $x = a$ or $\frac{dx}{dy} = \frac{m}{n}\frac{x}{y}$, therefore $y^m = Cx^n$.

6345. (By Prof. CASEY, F.R.S.)—A uniform circular plate is placed with its centre upon a prop; find at what points on its circumference three given weights must be attached in order that it may rest in a horizontal position.

Solution by D. EDWARDES; J. HAYASH, M.A.; *and others.*

If α, β, γ be the angles subtended at the centre by the weights in pairs; then, taking moments about a vertical plane through the centre and the weight w_3, we have $w_1 \sin\beta = w_2 \sin\alpha$; hence

$$w_1 : w_2 : w_3 = \sin\alpha : \sin\beta : \sin\gamma,$$

and α, β, γ are consequently the complements of the angles of a triangle whose sides are proportional to the weights.

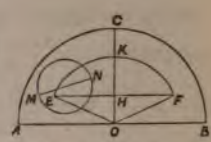
5592. (By J. DAWSON.)—Given one point in the circumference of a circle, and two points whose respective distances from the centre are the m^{th} and n^{th} parts of the radius; show how to draw the circle.

Solution by Prof. MOREL; Prof. MATZ, M.A.; *and others.*

Le centre de ce cercle se trouve sur deux lieux géométriques faciles à déterminer. Soit A le point sur la circonférence, B et C les deux autres points; on a, O étant le centre, OB : OA = 1 : m; donc le point O est sur une circonférence de cercle que l'on trace immédiatement. De même OC : AO = 1 : n; donc O se trouve sur une seconde circonférence de cercle. Il en résulte qu'il est à leur point de rencontre. On a ensuite le rayon en joignant le point O au point A.

5651. (By Prof. Seitz, M.A.)—If a circle be drawn on the line joining two points taken at random in the surface of a given semicircle; show that the chance that the circle lies wholly within the semicircle is $\frac{4}{3} - \frac{128}{45\pi}$.

Solution by the Proposer.



Let ABC be the given semicircle; O its centre; M, N the random points. With O as a centre, and a radius equal to the difference between the given radius and the half of MN, draw the arc EKF, the chord EF being at a distance from AB equal to the half MN; and draw OH perpendicular to EF. Now, while MN is given in length and direction, the area of the segment EKF represents the number of ways the two points can be taken, so that the circle on MN will be wholly within the semicircle.

Let $MN = 2x$, $AO = r$, θ = the angle which MN produced makes with AB. Then $OH = x$, $OE = r - x$, and

$$\text{area of segment EKF} = (r-x)^2 \cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}.$$

An element of the semicircle at N is $4x\,dx\,d\theta$; also the limits of x are 0 and $\frac{1}{2}r$; and of θ, 0 and 2π. The whole number of ways the two points can be taken is $\frac{1}{4}\pi^2 r^4$. Hence we have

$$p = \frac{4}{\pi^2 r^4}\int_0^{\frac{1}{2}r}\int_0^{2\pi}\left[(r-x)^2\cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}\right] 4x\,dx\,d\theta$$

$$= \frac{32}{\pi r^4}\int_0^{\frac{1}{2}r}\left[(r-x)^2\cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}\right] x\,dx = \frac{4}{3} - \frac{128}{45\pi}.$$

6615. (By H. McColl, B.A.)—Speaking of the limits of multiple integrals, if z_r (the r^{th} limit of z) be less than z_m and greater than z_n, we have the statemental equation $z_{m'n} = z_{m'r} + z_{r'n}$. In what sense is this equation true when z_r is not thus restricted?

Solution by W. B. Grove, B.A.; E. H. Haigh, B.A., B.Sc.; *and others.*

In $z_{m'.n} = z_{m'.r} + z_{r'.n}$ the letters denote statements as to the upper and lower limits of integration. Let $(z_{m'.n})$ denote the value of the integral taken between those limits, then the equation $(z_{m'.n}) = (z_{m'.r}) + (z_{r'.n})$ is universally true. For suppose $r < n$, then

$$(z_{r'.n}) = -(z_{n'.r}) \quad \text{and} \quad (z_{m'.r}) = (z_{m'.n}) + (z_{n'.r}),$$

therefore

$$(z_{m'.r}) + (z_{r'.n}) = (z_{m'.n}).$$

6613. (By G. S. Carr, B.A.)—If A, B, C are the normal, and F, G, H the tangential stresses upon a rectangular element of a strained elastic solid, show—(1) that the resultant tangential stress, at the same point, upon a plane whose direction-cosines with respect to the axes of A, B, C are l, m, n, is $[\{F(m^2-n^2)+(Gm-Hn)l-(B-C)mn\}^2+\&c.]^{\frac{1}{2}}$;

and (2) that its direction-cosines are proportional to

$$l\{A(m^2+n^2)-Bm^2-Cn^2-2mnF\}+(m^2+n^2-l^2)(Gn-Hm), \&c.$$

Solution by D. Edwardes; Lizzie A. Kittudge; *and others.*

(1) Let F' denote the stress in question, and let P, Q, R be the components, in the direction of the axes, of the stress on the plane; then the stress normal to the plane is $lP+mQ+nR$. Hence

$$P^2+Q^2+R^2=(lP+mQ+nR)^2+F'^2,$$

or
$$F'^2=(mR-nQ)^2+(nP-lR)^2+(lQ-mP)^2.$$

Now $P=Al+Hm+Gn$, $Q=lH+mB+nF$, $R=lG+mF+nC$,

in the small strain; wherefore, substituting in the above expression,

$$F'^2=[F(m^2-n^2)+(Gm-Hn)l-(B-C)mn]^2+\&c.+\&c.$$

(2) Let X be the x-component of the stress F'. Then, projecting along the axis of x, $P=l(lP+mQ+nR)+X$, or $X=P(m^2+n^2)-lmQ-nlR$; substituting for P, Q, R their values given above, we have

$$X=l[A(m^2+n^2)-Bm^2-Cn^2-2mnF]+(m^2+n^2-l^2)(mH+nG),$$

and similarly for the other components.

6577. (By the Rev. T. R. Terry, M.A.)—If a triangle PQR be inscribed in a parabola, so that PQ, PR are the normals at Q, R; show that (1) the centre of gravity of the triangle lies on the axis of the parabola; and (2) the side QR passes through a fixed point on the axis produced at a distance from the vertex equal to the semi-latus rectum.

Solution by E. W. Symons, M.A.; R. Knowles, B.A., L.C.P.; *and others.*

Q and R, the feet of normals from P (x', y'), are two of the intersections of the curves $y^2-4ax=0$, $xy+y(2a-x')-2ay'=0$, and also $y'^2=4ax'$. Eliminating x, we get a cubic for y with no square term, therefore centre of gravity of the three points P, Q, R lies on $y=0$.

Again, if $lx+my+n=0$ be equation of QR,

$$K(y^2-4ax)+(lx+my+n)(y-y')=0$$

can be made identical with $xy+y(2a-x')-2ay'=0$. This gives $K+m=0$, $l=\frac{n}{2a}$, &c. &c.; therefore QR is $n(x+2a)+2amy=0$, which passes through fixed point $(-2a, 0)$; therefore, &c.

6655. (By G. F. WALKER, M.A.)—Show that, according as q is an integer numerically greater than or less than p,

$$\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos^2 q\theta \, d\theta = \frac{\pi}{2^{2p+2}} \frac{(2p)!}{(p!)^2},$$

or

$$\frac{\pi}{2^{2p+2}} \left\{ \frac{(2p)!}{(p!)^2} + \frac{(2p)!}{(p+q)!\,(p-q)!} \right\}.$$

Solution by the Rev. T. R. TERRY, M.A.; Prof. MATZ, M.A.; *and others.*

$$\mathrm{I} = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta\,(1+\cos 2q\theta)\,d\theta = \tfrac{1}{2} \cdot \frac{(2p)!}{2^{2p}.p!\,p!}\tfrac{1}{2}\pi + \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta \, d\theta.$$

Now $2^{2p}\cos^{2p}\theta = 2\,[\cos 2p\theta + 2p\cos 2\,(p-1)\,\theta + \ldots]$.

If $q > p$, obviously $\cos 2q\theta$ cannot occur in this series,

therefore $$\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta \, d\theta = 0.$$

If $p > q$, then $\cos 2q\theta$ does occur, and its coefficient $= \dfrac{(2p)!}{(p-q)!\,(p+q)!}$.

Therefore $\displaystyle\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta \, d\theta = \frac{(2p)!}{(p-q)!\,(p+q)!} \cdot \frac{1}{2^{2p}} \cdot \tfrac{1}{2}\pi$; therefore, &c.

6262. (By the Rev. H. G. DAY, M.A.)—If a line of length c is divided into n parts, $x, y, z, \ldots$, by $(n-1)$ points taken at random in it, prove that the average values (1) of xyz, (2) of $x^\alpha y^\beta z^\gamma$ are

$$\frac{(n-1)!}{(2n-1)!}c^n \text{ and } \frac{(n-1)!\,\alpha!\,\beta!\,\gamma!\ldots}{(n-1+\alpha+\beta+\gamma+\ldots)!}c^{\alpha+\beta+\gamma+\ldots}$$

Solution by the PROPOSER.

Let $u_n c^n$ be the required quantity; also let x be the first part; hence, since the remainder is divided into $(n-1)$ parts at random, $u_n c^n$ = average value of $x\,(c-x)^{n-1}u_{n-1}$, x being taken according to conditions prescribed;

therefore $$u_n c^n = \int_0^c x u_{n-1}\,(c-x)^{n-1}\left(\frac{c-x}{c}\right)^{n-2}(n-1)\,\frac{dx}{c},$$

$$u_n = \int_0^1 (n-1)\,x\,(1-x)^{2n-3}\,dx\,u^{n-1} = \frac{(n-1)\,u_{n-1}}{(2n-1)\,(2n-2)} = \frac{(n-1)!}{(2n-1)!}.$$

2. If $x, y, z \ldots$ are the successive parts, the average value of

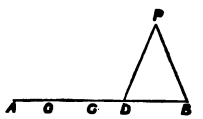
$$x^\alpha y^\beta z^\gamma \ldots \text{ is } \frac{(n-1)!\,\alpha!\,\beta!\,\gamma!\ldots}{(n-1+\alpha+\beta+\gamma+\ldots)!}c^{\alpha+\beta+\gamma+\ldots}.$$

[Mr. DAY remarks that, though he has more than once been on the point of admitting a fallacious Solution, he can guarantee the correctness of the values of the chances given above.]

5405. (By S. Tebay, B.A.)—A straight rod, resting on a smooth horizontal plane, is struck by a random shot from a given point in the plane; find the mean angular velocity of the rod, and the mean motion of the centre of gravity of the rod.

Solution by the Proposer.

Let AB be the rod, bisected in C, P the point from which the shot is fired, D the point of impact, and O the instantaneous centre of rotation. Let $AB = 2a$, $BP = b$, $CD = h$, $OC = x$, $\angle ABP = \alpha$, $BPD = \theta$, ω the angular velocity of the rod, m its mass, m' the mass of the shot, u its velocity, and F the effect of the blow at D. Then we have



$$(x^2 + \tfrac{1}{3}a^2)\, m\omega = F(x+h), \quad mx\omega = F, \quad \text{from which} \quad \tfrac{1}{3}a^2 m\omega = Fh.$$

But
$$h = a - \frac{b\sin\theta}{\sin(\alpha+\theta)}, \quad F = m'u\sin(\alpha+\theta);$$

therefore
$$\tfrac{1}{3}a^2 m\omega = m'u\{a\sin(\alpha+\theta) - b\sin\theta\}.$$

If the shot be fired between the limits $\theta = \gamma$, $\theta = \beta$, the mean angular velocity
$$= \frac{1}{\beta-\gamma}\int_\gamma^\beta \omega\, d\theta$$
$$= \frac{3m'u}{a^2 m(\beta-\gamma)}\sin\tfrac{1}{2}(\beta-\gamma)\{a\sin\tfrac{1}{2}(2\alpha+\beta+\gamma) - b\sin\tfrac{1}{2}(2\alpha+\beta+\gamma)\}.$$

Let V be the velocity of the centre of gravity; then
$$V = \frac{F}{m} = \frac{m'u}{m}\sin(\alpha+\theta);$$

therefore
$$\frac{1}{\beta-\gamma}\int_\gamma^\beta V d\theta = \frac{m'u}{m(\beta-\gamma)}\{\cos(\alpha+\gamma) - \cos(\alpha+\beta)\}.$$

6696. (By D. Edwardes.)—If $m\sin(\theta+\phi) = \cos(\theta-\phi)$, prove that $(1-m\sin 2\theta)^{-1} + (1-m\sin 2\phi)^{-1} = 2(1-m^2)^{-1}$.

Solution by Prof. Scott, M.A.; J. O'Regan; *and others.*

If $m\sin(\theta+\phi) = \cos(\theta-\phi)$, then $\tan\phi = \dfrac{1-m\tan\theta}{m-\tan\theta}$; thence

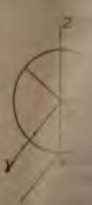
$$\sin 2\phi = \frac{2m-(1+m^2)\sin 2\theta}{1+m^2-2m\sin 2\theta}, \text{ and } (1-m\sin 2\phi)^{-1} = \frac{1+m^2-2m\sin 2\theta}{(1-m^2)(1-m\sin 2\theta)},$$

which gives
$$(1-m\sin 2\theta)^{-1} + (1-m\sin 2\phi)^{-1} = \frac{2}{1-m^2}.$$

6459. (By C. Leudesdorf, M.A.)—A homogenous sphere of mass m and radius a is rotating about a diameter with angular velocity Ω whil its centre is moving in the direction of this diame'er with velocity V when it strikes a perfectly rough horizontal plane. Show that the kine energy lost by the impact is $\frac{1}{2}mV^2(1-e^2)\cos^2\theta+\frac{1}{7}m(V^2+a^2\Omega^2)\sin$ θ being the angle of incidence, and e the coefficient of restitution.

Solution by G. S. Carr, B.A.; D. Edwardes; *and others.*

Take rectangular axes as in the figure, the plane XZ to cont axis of the rotating sphere, and also its point of contact A with th plane.



The first term is the *vis viva* lost through imperfect elasticity, and is a familiar result. The second expresses the amount lost through the tangential reaction at A. To find this, let u be the velocity of the point A resolved parallel to the plane, and which is wholly destroyed by the impact. By resolving V and Ω along the X and Y axes, we get

$$u^2 = V^2\sin^2\theta+a^2\Omega^2\sin^2\theta.$$

Now, if an impulse P applied tangentially to a sphere generates a velocity u at the surface, v at the centre, and an angular velocity ω, we know from the equatio $P = mv$, and $mk^2\omega = Pa$, that $v = \frac{2}{7}u$ and $a\omega = \frac{5}{7}u$. Hen generated is $\frac{1}{2}m(v^2+k^2\omega^2) = \frac{1}{7}mu^2$. This is therefore the the present case, by destroying the velocity u.

6583. (By W. J. C. Sharp, M.A.)—Show th $\left(\frac{1+p^2}{q}\frac{d}{dx}\right)^n \cdot \frac{(1+p^2)^{\frac{3}{2}}}{q} = 0$ represents any $(n-1)^{\text{th}}$ inv

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Solution by E. W. Symons, M.A.; J

If $\rho = f(\phi)$ be equation to a curve,

angle which tangent makes with a fi

curvature of corresponding poin

of curvatur on n^{th} e
volute of a n^{th} evol
radius of cu of a poi

lute is $\frac{d^n\rho}{d\phi^n}$

or.

6575. (By H. McColl, B.A.)—Let S denote any multiple integral; αS the value of S when the integration is restricted by the statement α; $(\alpha-\beta+\gamma-\ldots)$ S an abbreviation for $\alpha S-\beta S+\gamma S-\ldots$; and $\alpha(1-\beta)(1-\gamma)S$ an abbreviation for $(\alpha-\alpha\beta-\alpha\gamma+\alpha\beta\gamma)$ S, and so on; then show that $\alpha\beta'\gamma'\ldots S=\alpha(1-\beta)(1-\gamma)\ldots S$. [Mr. McColl states that this simple theorem will often effect great simplification in dealing with the limits of multiple integrals].

Solution by W. B. Grove, B.A.

Since $\alpha S=\alpha(\beta+\beta')(\gamma+\gamma')S=(\alpha\beta\gamma+\alpha\beta\gamma'+\alpha\beta'\gamma+\alpha\beta'\gamma')S$,

$\therefore\ \alpha\beta'\gamma'S=(\alpha-\alpha\beta-\alpha\beta'\gamma)S=(\alpha-\alpha\beta-\alpha\gamma+\alpha\beta\gamma)S=\alpha(1-\beta)(1-\gamma)S$.

In fact, these integrals can be treated in exactly the same way as the combinations in Prof. Jevons' *Numerical Logic*, in which

$$(ab'c')=(a)-(ab)-(ac)+(abc).$$

6528. (By R. B. Hayward, M.A.)—In a "three-cornered" constituency (*i.e.*, one which returns three members) each voter has two votes, but cannot give both to the same candidate. Supposing the majority, consisting of M voters, to put forward three candidates, and the minority, consisting of m voters, to put forward two, and supposing that all the voters take part in the election and give both their votes; prove that (1) the chance that the three candidates of the majority will be all elected is $1-3\frac{(m+1)(m+2)}{(M+1)(M+2)}$ or $\frac{(2M-3m-1)(2M-3m-2)}{(M+1)(M+2)}$, according as m is $<$ or $>\frac{1}{2}M$, and $=\frac{1}{4}$ nearly if $m=\frac{1}{2}M$; and (2) that, if $m=\frac{2}{3}M$, the chance that two of the three will be defeated is $\frac{1}{18}$ nearly.

Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others.*

Let A, B, C be the majority candidates, and let x denote the number who vote for B and C, y ... C and A, z ... A and B. Then $x+y+z=M$. Also, if A, B, and C are to be elected, $y+z=>m$, $z+x=>m$, $x+y=>m$

Take an equilateral triangle of reference, and trace the lines

$$(y+z)M=m(x+y+z),\ \&c.$$

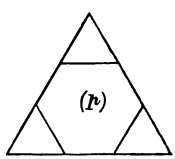


Fig 1.

If $M>2m$, we have Fig. 1, and if $M<2m$, we have Fig. 2. The required chance is the number of points which fall in area (p) in either figure.

In the first case, it is best to find number in either of the smaller triangles. Thus, as

$$1+2+\ \ldots\ m+1=\tfrac{1}{2}(m+1)(m+2),$$

the chance is

$$1-\left\{3\left[\tfrac{1}{2}(m+1)(m+2)\right]\div\tfrac{1}{2}(M+1)(M+2)\right\}$$

$$=1-3\frac{(m+1)(m+2)}{(M+1)(M+2)}.$$

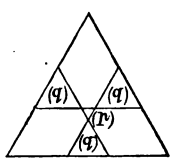


Fig. 2.

In the second case, the number is equal to

$$1+2+\ldots[(m+1)-2(m-M)] = \tfrac{1}{2}(2M-3m+1)(2M-3m+2),$$

and the required chance is $\dfrac{(2M-3m+1)(2M-3m+2)}{(M+1)(M+2)}$.

The chance that two of the three are rejected, is the sum of the number of points in the areas (q), to whole numbers, and the chance is

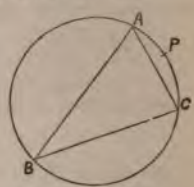
$$3\frac{(2m-M+1)(2m-M+2)}{(M+1)(M+2)}.$$

The special chances are easily found by substituting for m, and supposing M large, as would usually be the case in such a constituency.

6688. (By C. Leudesdorf, M.A.)—If a, b, c be three quantities such that any two are together greater than the third, and if x, y, z be three quantities whose sum is positive; show that, if $a^2x^{-1}+b^2y^{-1}+c^2z^{-1}=0$, the product xyz must be negative.

I. *Solution by* E. W. Symons, M.A.; J. O'Regan; *and others.*

a, b, c are the sides of a triangle; x, y, z are proportional to the triangular coordinates of a point with respect to it; and because $a^2x^{-1}+\ldots=0$, this point lies on the circumscribed circle of the triangle; therefore the point (x, y, z) lies outside the triangle, and within one of its angles; and therefore one of the three x, y, z is negative; therefore &c.



II. *Solution by* R. Tucker, M.A.; A. Martin, M.A.; *and others.*

$$x+y+z=k \text{ (positive)}:$$

therefore $\quad 0 = a^2y(k-x-y)+b^2x(k-x-y)+c^2xy,$

i.e., $\quad (a^2y+b^2x)k = (a^2+b^2-c^2)xy+a^2y^2+b^2x^2;$

therefore $\quad -c^2xyk = [(a+b)^2-c^2]xyz+(ay-bx)^2z$

or $\quad = (ay+bx)^2z-[c^2-(a-b)^2]xyz\ldots\ldots\ldots\ldots(1, 2).$

From (1), if xy be positive, z cannot be positive; from (2), if xy be negative, z cannot be negative; therefore xyz is negative.

6406. (By Prof. Townsend, F.R.S.)—The equation of a conic in terms of the three perpendiculars λ, μ, ν on a variable tangent from the three vertices A, B, C of a fixed triangle self-reciprocal with respect to the curve,

being given in the form $l\lambda^2 + m\mu^2 + n\nu^2 = 0$, where l, m, n are constants; find, in terms of the same coordinates, that of the point-pair in which the axes of the curve intersect the line at infinity.

Solution by the PROPOSER.

Since evidently, for every diameter of the curve, $l\lambda + m\mu + n\nu = 0$, which accordingly is the tangential equation of its centre, and since consequently, for any arbitrary line in its plane, the perpendicular distance from its centre $= (l\lambda + m\mu + n\nu) \div (l + m + n)$, we have therefore to find the relation of the second order connecting λ, μ, ν, for which the sum $(l\lambda + m\mu + n\nu)$ shall have its greatest and least values, subject to the given condition of tangency with the conic, viz. $l\lambda^2 + m\mu^2 + n\nu^2 = 0$, and to the general relation connecting the three perpendiculars from the three vertices of the triangle upon any arbitrary line in the plane, viz.

$$a^2 . \lambda^2 + b^2 . \mu^2 + c^2 . \nu^2 - 2bc \cos \alpha . \mu\nu - 2ca \cos \beta . \nu\lambda - 2ab \cos \gamma . \lambda\mu = 4\Delta^2,$$

where a, b, c are the three sides, α, β, γ the three angles, and Δ the area of the triangle; hence, differentiating with respect to the three variables, the sum and the two equations, and eliminating the three differentials, we get at once the determinantal equation

$$a (a . \lambda - c \cos \beta . \nu - b \cos \gamma . \mu) \; mn (\mu - \nu) + \&c. = 0,$$

which accordingly is that of the point-pair in question.

Arranged in powers and products of the variables, this equation becomes

$$p\lambda^2 + q\mu^2 + r\nu^2 - 2(s-p)\mu\nu - 2(s-q)\nu\lambda - 2(s-r)\lambda\mu = 0,$$

where $al (cm \cos \beta - bn \cos \gamma) = p$, &c., and $p + q + r = 2s$; from which it appears that the point-pair in question is, as it ought to be, harmonically conjugate with respect to the point-pair

$$a^2 . \lambda^2 + b^2 . \mu^2 + c^2 . \nu^2 - 2bc \cos \alpha . \mu\nu - 2ca \cos \beta . \nu\lambda - 2ab \cos \gamma . \lambda\mu = 0,$$

which represents, to the same coordinates, the two circular points at infinity.

When $l : m : n = a \sec \alpha : b \sec \beta : c \sec \gamma$, that is, when the centre of the conic is the orthocentre of the triangle, and when the conic itself is consequently the orthocentric circle of the triangle, the preceding equation becomes, as it ought, indeterminate; every line through its centre being then an axis of the curve.

6502 & 6589. (By S. TEBAY, B.A.)—(6502). The letters in the annexed square represent any 16 numbers in arithmetical progression, the sum of the two extremes being s. They are so arranged that $2s$ is the sum of each of the 24 groups *abcd*, *efgh*, *ijkl*, *mnpq*, *aeim*, *bfjn*, *cgkp*, *ahlq*, *afkq*, *dgjm*, *abef*, *begf*, *cdgh*, *efij*, *fgjk*, *ghkl*, *klpq*, *ijmn*, *jknp*, *acik*, *bdjl*, *egmp*, *fhnq*, *admq*.

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*l*
m	*n*	*p*	*q*

Find in how many essentially different ways the numbers can be thus arranged, exclusive of all reversions, such as *dcba*, *miea*, &c.; and also the total number of ways in which four of these numbers make up $2s$.

(6589). The letters in the annexed square represent the first sixteen consecutive numbers, so arranged that each of the sixteen groups *abef*,

edjh, ijmn, klpq, fgjk, admq, abcd, efgh, ijkl, mnpq, aeim, bfjn, cgkp, dhlq, afkq, dgjm makes 34. There are 432 essentially different arrangements of these numbers, but a demonstration of the following property, with other allied properties, is still a desideratum. If $a+c=17$, prove that $e+g=17$.

Solution by G. HEPPEL, M.A.

Answering, first of all, the last part of Quest. 6502, there are 86 different sets of four numbers making up $2s$. The progression being $u+v$, $u+2v$, &c. ... $u+16v$, it is easy to see that, by taking u from each, and dividing by v, the case reduces to that of the first 16 natural numbers. Of these there are 19 of the required sets where 1 is the least number, 19 where 2 is the least, 18 where 3, 15 where 4, 10 where 5, 4 where 6, and 1 where 7. This gives a total of 86.

The case in Quest. 6502 is included in the more general Quest. 6589. Considering the latter, the following preliminary properties must be proved.

$a+f+k+q=34$, $k+q+p+l=34$, $\therefore a+f=p+l$, and $b+e+p+l=34$,

$a+b+c+d=34$, $a+d+m+q=34$, $\therefore b+c=m+q$,

and $b+c+n+p=34$. Hence also $e+l=c+n$,

$e+f+g+h=34$, $f+g+j+k=34$, $\therefore e+h=j+k$,

$a+b+c+d=34$, $c+d+g+h=34$, $\therefore a+b=g+h$,

$a+m+d+q=34$, $m+j+g+d=34$, $\therefore a+q=j+g$,

$a+q=j+g$, and $i+j=p+q$, $\therefore a+i=g+p$,

$\therefore a+i+c+k=34$, $i+c=f+q$, $a+k=h+n$.

These equalities are merely types of several similar ones of each kind.

Call a set of four of the numbers necessarily making up 34 a *range*. Let two numbers of such a set be said to be in range with one another. Then, first it is necessary to prove that, if two numbers in range make up 17, there are eight pairs symmetrically situated each making up 17. The several cases must be considered separately.

First, let $a+b=17$. This pairs off a, b, c, d, e, f, g, h.
If $j+i=17$ there is symmetry.

$j+k=17$ gives $e+h=17$,
$j+l=17$,, $b+d=17$,
$j+m=17$,, $c+h=17$,
$j+n=17$,, $b+f=17$,
} all inconsistent with the hypothesis.

If $j+p=17$, the only possibility for i is $i+q=17$.
If $j+q=17$, ,, ,, $i+p=17$.
In either case $i+j+p+q=34$, and $i+j=p+q$; therefore $i+j=17$, which is inconsistent; therefore there must be symmetry.

Secondly, let $a+f=17$. This pairs off a, f, b, e, k, l, p, q. If $d+g=17$ there is symmetry. $d+c=17$, $d+h=17$, $d+m=17$, $d+j=17$, lead to impossibilities. $d+i=17$ and $d+n=17$ are similarly situated, so that one only need be considered. Suppose $d+i=17$, then the only pair for n is $n+g=17$, for $n+c=17$ gives $b+p=17$. Hence $d+i+n+g=34$, and $d+g=i+n$; therefore $d+g=17$, which is inconsistent.

Thirdly, let $a+k=17$. This pairs off a, k, c, i, f, q, h, n. If $d+j=17$, there is symmetry. If not, $d+e=17$ and $d+p=17$ are alone possible and are similar. Suppose $d+e=17$, then must $j+p=17$; therefore $d+e+j+p=34$, $d+j=e+p$; therefore $d+j=17$, which is inconsistent.

Fourthly, let $a+d=17$. This pairs off a, d, b, c, m, q, n, p. If $e+h=17$, there is symmetry. If not, we must have either $e+k=17$, $h+j=17$, or $e+j=17$, $h+k=17$; whence $j+k=17$, which is inconsistent.

Fifthly, let $a+c=17$. This pairs off a, b, c, d, i, j, k, l. If $m+p=17$, there is symmetry. If not, then either $m+f=17$, $p+h=17$, or $m+h=17$, $p+f=17$; whence $m+p=17$, which is inconsistent.

Sixthly and lastly, let $a+q=17$. This pairs off a, q, d, m, f, k, g, j, and we may have the symmetrical combination $a+q=17$, $m+d=17$, $f+k=17$, $g+j=17$, $b+p=17$, $c+n=17$, $e+l=17$, $i+h=17$. If not, then must either $e+c=17$, $n+l=17$, or $e+n=17$, $c+l=17$. This leads to four symmetrical arrangements, two of which it will be sufficient to consider, as the remaining two are similar to them.

I. $e+c=17$, $b+i=17$, $n+l=17$, $p+h=17$.

II. $e+c=17$, $b+h=17$, $n+l=17$, $p+i=17$.

In case I. $a+b+c+e+f+g+i+j+k=68+a$,

$d+h+l+q+m+n+p=68-q$.

Therefore, adding, $136+a-q=136$, therefore $a=q$, which is impossible.

In case II. $e+b+c+h=34$, therefore $b+c=f+g$;

therefore $b+c+j+k=34$, therefore $a+i=b+j$.

Now $e+c+i+p=34$, therefore $e+i=g+k$;

therefore $e+i+f+j=34$, therefore $a+b=i+j$.

But it was shown that $a+i=b+j$, therefore $a=j$, which is impossible; therefore $a+q=17$ gives only the symmetrical arrangements first mentioned.

Every range is one of a set of four independent ranges together comprising the whole 16 numbers. Thus *bepl* is one of the set *bepl, chin, afkq, dgjm*.

Every range must contain two even and two odd numbers. For, if one range be all odd, another must be all even, and *vice versâ*. The remaining two independent ranges cannot be all odd and all even respectively, for then the sum of the two even ranges would be 72, and of the two odd ranges 64, which is impossible. And if the remaining two consist each of two odd and two even, it will be found impossible to place them so as to make the sums of all the ranges even.

Every pair of numbers in a range must produce a sum which is repeated one or more times. For instance, $a+b=g+h$, $a+f=p+l$. It follows from this that both 2 and 3 must be out of range with 1, for the remaining 31 and 30 can each be made up only in one way. Similarly 15 and 14 must both be out of range with 16.

Every number must form a part of *six* distinct ranges. Thus k is in *cgkp, ijkl, afkq, acik, fgjk, klpq*.

Every pair in a range occur again in company in one and only one more range. Thus c, h occur in *cdgh* and *chin*; d, j occur in *dgjm* and *dbjl*.

There are nineteen sets of three numbers which in range with 1 make up 34. From these we have to find six sets such that every number occurs in two and only two sets. From the list must be excluded 16 15 2, 16 14 3 as impossible, and 15 13 5, 15 11, 7, 13 11 9 as being all odd.

First, attempting to proceed without using 16, 15 14 4 is impossible, 15 12 6, 15 10 8 require 14 13 6. This involves 13 12 8, and no second 14 is possible, for 14 12 7 gives 7 only once; 14 11 8 gives 8 three times. Hence 1 must be in range with 16, and any arrangement of the square must be one of the six symmetrical forms.

Every range must consist of two numbers greater than 8, and two not greater than 8. For let $2s = 17$, and let $2t$, $2u$, $2v$, $2w$ represent different odd numbers not greater than 15. Let the top row of the square, if possible, be

$$s-t \quad s-u \quad s-v \quad s+t+u+v,$$

giving three numbers not greater than 8. Suppose that the square falls under the fifth kind of symmetry, where $a+i = 17$. Fill up e by $s+w$, and complete the square, when the scheme will stand thus:—

$s-t$	$s-u$	$s-v$	$s+t+u+v$
$s+w$	$s+t+u-w$	$s-t-2u$ $-v+w$	$s+u+v-w$
$s+t$	$s+u$	$s+v$	$s-t-u-v$
$s-w$	$s-t-u+w$	$s+t+2u$ $+v-w$	$s-u-v+w$

If

$a=f$, $w=2t+u$; $a=g$, $w=2u+v$; $a=h$, $w=t+u+v$;
$a=p$, $w=2t+2u+v$; $a=q$, $w=u+v-t$;
$b=f$, $w=t+2u$; $b=p$, $w=t+3u+v$; $c=h$, $w=u+2v$;
$c=n$, $w=t+u-v$; $c=p$, $w=t+2u+2v$; $d=g$, $w=2t+3u+2v$;
$e=f$, $2w=t+u$; $e=h$, $2w=u+v$; $e=p$, $2w=t+2u+v$;
$f=g$, $2w=2t+3u+v$; $g=h$, $2w=t+3u+2v$; $g=p$, $w=t+2u+v$;

and the converses of these (which are going to be used) are also true. The other equalities give either manifest impossibilities or useless repetitions. Now $t+u+v$ must be less than s, or else l would be impossible. The only values, therefore, for $2t$, $2u$, $2v$ are:—

1	3	5	leaving	7	9	11	13	15	for $2w$
1	3	7	,,	5	9	11	13	15	,,
1	3	9	,,	5	7	11	13	15	,,
1	3	11	,,	5	7	9	13	15	,,
1	5	7	,,	3	9	11	13	15	,,
1	5	9	,,	3	7	11	13	15	,,
3	5	7	,,	1	9	11	13	15	,,

and every one of these values of $2w$ fulfils one of the equations given above, renders two elements of the square equal, and thus involves an impossibility. The same reasoning applies if any other form of symmetry be chosen, for the same terms will appear in the scheme, only differently placed. Also, if the top row had been $s+t$, $s+u$, &c., or if $s-w$ had been placed at e, the same equations would have reappeared.

The list of numbers making a range with 1 must now be further corrected by striking out 14 10 9, 12 11 10 as consisting of numbers greater than 8, and then striking out 16 12 5, 16 9 8 as containing numbers not repeated. The list then becomes

16	13	4		15	14	4		14	13	6	13 12 8
16	11	6		15	12	6		14	12	7	
16	10	7		15	10	8		14	11	8	

Now for the six ranges including 1 there must be two with 16, and there are three sets which may easily be filled up, thus—

16	13	4	16	13	4	16	11	6
16	11	6	16	10	7	16	10	7
15	14	4	15	14	4	15	12	6
15	12	6	15	10	8	15	10	8
14	11	8	14	12	7	14	12	7
13	12	8	13	12	8	14	11	8

By taking the differences of these from 17, we get the ranges of which 16 is a member, namely—

1	4	13	1	4	13	1	6	11
1	6	11	1	7	10	1	7	10
2	3	13	2	3	13	2	5	11
2	5	11	2	7	9	2	7	9
3	6	9	3	5	10	3	5	10
4	5	9	4	5	9	3	6	9

We are now able to form any square, and to tell the number of arrangements. First, place 1 in any of the 16 positions. Secondly, put 16 in any one of the 9 places in range with 1. Thirdly, choose any one of the three possible pairs of ranges with 1 and 16. Fourthly, place in range the numbers so chosen. This can be done in 8 ways, as will be seen by the example to be given presently. The square can now be completed in only one way, which is determined by considering the necessary ranges of which 16 forms an element. Thus there are $16 \times 9 \times 3 \times 8$, or 3456 variations. But these include repetitions, such as may be made by turning the square round, bringing each side in succession to the top, and by reversing the order of the rows or columns. These two causes bring about eight variations for each distinct set. There are, therefore, 432 essentially different arrangements.

Example.—Place 1 at *j*. Place 16 at *l*, giving the fifth symmetrical system. Chose the 16 13 4, 16 10 7 set of ranges. Then we obtain the *eight* partly formed squares—

.	10	.	7	.	7	.	10	.	10	.	7	.	7	.	10
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
13	1	4	16	13	1	4	16	4	1	13	16	4	1	13	16
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	13	.	4	.	4	.	13	.	13	.	4	.	4	.	13
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
10	1	7	16	10	1	7	16	7	1	10	16	7	1	10	16
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

To complete these, remember that 2 must be in range with 16, 7 and 16, 13. This gives as its places in the above eight squares, *h*, *e*, *q*, *p*, *p*, *q*, *e*, *h*. The place of 15 is then determined, and the square easily filled.

There is a simpler practical method of forming the squares, which requires no knowledge of the range numbers, but is not suited to furnish a plan for a demonstration.

RULES.

1. Choose any place for 1.
2. Choose any of the *nine* available places for 16. This determines the nature of the symmetry.
3. Take for 2 any of the six places not in range with 1, provided that the consequent place of 15 is not out of range with 1. There will always be *four* of the six places that will suit, and two that will not.

4. Take for 3 one of the remaining places out of range with 1, and not involving any immediately apparent impossibility. There will always be *three* places suitable. Remember that 2, 3 must not form part of an open range.

5. 4 and 13 will have their places determined by the above operations.

6. There are then always *two* and only two places for 5. These may be more easily found by remembering that 5 cannot go in a range with two numbers making up 14, 15, or 16, as the range could not be completed.

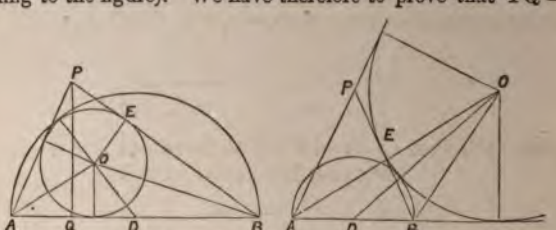
Number of arrangements $16 \times 9 \times 4 \times 3 \times 2 = 3456$ as before.

Question 6502 gives the particular case of the third system of symmetry, $a + k = 16$. There are, therefore, $432 \div 9$ or 48 essentially different arrangements.

6572. (By the Editor.)—If from the ends of a diameter AB of a circle, tangents AP, BP be drawn to a circle that touches the given circle and its diameter AB, prove that the sum or the difference of AP and BP will be equal to the sum or the difference of AB and the perpendicular PQ thereon from P, according as the contact of the tangential circle with the given circle and its diameter AB is internal or external.

Solution by W. B. Grove, B.A.; D. Eastwood, M.A.; *and others.*

Let PB meet the tangential circle at E; then $AP \pm BP = AB \pm 2PE$ (according to the figure). We have therefore to prove that $PQ = 2PE$.



Let $AB = c$, radius of tangential circle $(O) = r$; then we have

$$(\tfrac{1}{2}c - r)^2 = OD^2 = r^2 \operatorname{cosec}^2 \tfrac{1}{2}A + \tfrac{1}{4}c^2 - cr \cot \tfrac{1}{2}A, \quad \therefore \quad \frac{r}{c} = \frac{1 - \tan \tfrac{1}{2}A}{\cot \tfrac{1}{2}A} \ldots\ldots (1, 2).$$

But $\dfrac{c}{r} = \cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B$, therefore $\dfrac{r}{c} = \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \ldots\ldots\ldots (3, 4)$;

therefore $\tfrac{1}{2}PQ = \dfrac{rs}{c} = s \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = s - c = s_3 = PE.$

In the second case, $\tfrac{1}{2}PQ = s - b = s_2 = PE$. Therefore, &c.

[In the foregoing solution, (2) × (3) gives

$$\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B \equiv \tan OAB + \tan OBA = 1,$$

a characteristic property of the tangency which is, in fact, needed to reduce (4), wherefrom we obtain (in Fig. 1)

$$1 = \frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} + \&c. = \frac{1-\cos A}{\sin A} + \&c. = \operatorname{cosec} A - \cot A + \&c.,$$

or $PQ(\operatorname{cosec} A + \operatorname{cosec} B) = PQ(1 + \cot A + \cot B)$, or $PA + PB = PQ + AB$, which furnishes, in this way, another proof of the theorem.

The locus of O is a parabola, and the locus of P a cubic; for which and other properties see Question 6834].

6253. (By B. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle.

Solution by W. H. Besant, M.A., F.R.S.

If α, β, γ be the perpendiculars from any fixed point on the sides a, b, c of the triangle formed by the lines, then $a\alpha + b\beta + c\gamma$ is constant; therefore, since $p + \frac{d^2p}{d^2\phi}$ is the radius of curvature of the envelop of a straight line, $a\rho_1 + b\rho_2 + c\rho_3$ is constant. The intrinsic equation of an involute of a circle is of the form $s = c\phi^2 + d\phi + e$, and it follows that, if ρ_1 and ρ_2 are each of the form $c\phi + d$, ρ_3 is of the same form.

The method of this solution, which is due to Mr. Ferrers, is given in my *Notes on Roulettes and Glissettes*.

[Another solution is given in *Reprint*, Vol. XXXIII., p. 67.]

6738. (By Prof. Crofton, F.R.S.)—Prove that

$$D^n f(xD)\,X = f(xD + n)\,D^n X.$$

Solution by J. J. Walker, M.A.

Whatever $\phi(D)$ and $\chi(D)$ may be, then, by a Theorem recently communicated by me to the Mathematical Society (Proc. Vol. XII., p. 197), we have

$$f[x\phi(D)]\chi(D)\,X = \chi(D) f[x\phi(D)]\,X - \phi(D)\frac{d\chi(D)}{dD} f'[\ldots] + \frac{1}{1.2}\left[\phi(D)\frac{d}{dD}\right]^2 \chi(D) f''[\ldots]\ldots$$

Now, if $\chi(D) = D^n$, $\phi(D) = D$, this gives $f(xD)\,D^n X = D^n f(xD - n)\,X$,

$$\therefore f(xD + n)\,D^n X = D^n\left\{f(xD - n) + nf'(xD - n) + \frac{n^2}{1.2} f''(xD - n) + \ldots\right\} X = D^n f(xD)\,X.$$

6766. (By Professor CAYLEY, F.R.S.)—Find the stationary and double tangents of the curve $x^4+y^4+z^4=0$.

Solution by the PROPOSER.

Take l a fourth root of -1; m and n fourth roots of $+1$; then the 28 double tangents are the lines $x=ly$, $x=lz$, $y=lz$, 4, $(4+4+4=)$ 12 lines; and the lines $x+my+nz=0$, 16 lines; and the first 12 of these, each counted twice, are the 24 stationary tangents. In fact, any one of the 12 lines is an osculating tangent, or line meeting the curve in 4 coincident points; it counts therefore once as a double tangent, and twice as a stationary tangent. There should consequently be 16 other double tangents; and it only needs to be shown that these are the 16 lines $x+my+nz=0$. Consider any one line $x+my+nz=0$; for its intersections with the curve $x^4+y^4+z^4=0$, we have

$$(my+nz)^4+y^4+z^4=0,$$

or, as this may be written, $(my+nz)^4+m^4y^4+n^4z^4=0$;

viz., this is $2\left(1, 2, 3, 2, 1 \mid my, nz\right)^4=0$,

or, what is the same thing, $2\left[\left(1, 1, 1 \mid my, nz\right)^2\right]^2=0$:

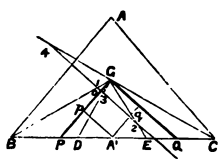
so that the line is a double tangent, the two points of contact being given by means of the equation $\left(1, 1, 1 \mid my, nz\right)^2=0$; viz., ω being an imaginary cube root of unity, we have $nz=\omega my$ or $\omega^2 my$: and thence, for the points of contact, $x:y:z=1:\frac{\omega}{m}:\frac{\omega^2}{n}$, or $=1:\frac{\omega^2}{m}:\frac{\omega}{n}$;

values which satisfy, as they should do, the two equations

$$x+my+nz=0 \text{ and } x^4+y^4+z^4=0.$$

5527. (By W. S. B. WOOLHOUSE, F.R.A.S.)—Determine, by a simple construction, the size of the equilateral triangle from or into which a triangle can be orthogonally projected.

Solution by C. LEUDESDORF, M.A.



Let ABC be a triangle, G its centre of gravity; and let BC be bisected at A′ and trisected at D and E; join G to B, D, E, C. In order to project ABC into a equilateral triangle, it will be sufficient to project so that BGE, DGC both become right angles. Draw with centres D and E two circles, each of radius DE, and through G and the points where these circles cut, draw a third circle cutting BC in P and Q. Then GP, GQ are the right-angled pair of the involution of which GB, GE; GD, GE are pairs; so that, if a transversal be drawn parallel either to GP or GQ, as 41032 in the figure, the point O will be the centre of the involution 12, 34, and

therefore semicircles described on 12, 34 as diameters, on the side nearer to G, will cut OG in the same point g. Taking g as the orthogonal plane projection of G, and the transversal as the axis, the projection of any point is at once found by drawing an ordinate to the transversal, and diminishing it in the ratio Og : OG ; and the angles BGE, DGC will both become right angles, and ABC an equilateral triangle. By drawing a transversal parallel to GP, we obtain another projection, which gives another equilateral triangle, that from which ABC can be orthogonally projected. The magnitude of the triangles may be found without actually constructing them. For GP, GQ are the axes of the ellipse touching the sides of ABC at their middle points; so that, if $A'p$, $A'q$ be drawn at right angles to GP, GQ respectively, the squares of the semi-axes are equal to $Gp \,.\, GP$ and $Gq \,.\, GQ$. If, then, circles be drawn on A'P and A'Q as diameters, the tangents to them from G will be equal in length to the radii of the circles which can be inscribed in the two equilateral triangles.

[Other solutions are given in the *Reprints*, Vol. XXIX., p. 83, and Vol. XXXI., p. 18.]

6411. (By Professor WOLSTENHOLME, M.A.)—If

$$y = x^n - nx^{n-2} + \frac{n(n-3)}{2} x^{n-4} - \frac{n(n-4)(n-5)}{3!} x^{n-6} + \ldots,$$

where n is an odd integer $(2r+1)$, prove that $\dfrac{y-2}{x-2} = (u_r + u_{r-1})^2$,

u_r being $\quad x^r - (r-1)\, x^{r-2} + \dfrac{(r-2)(r-3)}{2!} x^{r-4} - \ldots$.

Solution by J. HAMMOND, M.A. ; G. HEPPEL, M.A. ; *and others.*

Let $x = 2\cos\theta$; then $y = 2\cos n\theta$ and $u_r = \dfrac{\sin(r+1)\theta}{\sin\theta}$,

therefore $\qquad \dfrac{y-2}{x-2} = \dfrac{1-\cos n\theta}{1-\cos\theta} = \dfrac{\sin^2 \frac{1}{2} n\theta}{\sin^2 \frac{1}{2}\theta}$,

and

$$u_r + u_{r-1} = \frac{\sin(r+1)\theta + \sin r\theta}{\sin\theta} = \frac{2\cos\frac{1}{2}\theta \sin\frac{1}{2}(2r+1)\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \frac{\sin\frac{1}{2}(2r+1)\theta}{\sin\frac{1}{2}\theta},$$

therefore $\qquad \dfrac{y-2}{x-2} = (u_r + u_{r-1})^2$, if $n = 2r+1$.

5938. (By W. H. H. HUDSON, M.A.) — A sphere touches another sphere internally, find the centre of gravity of the mass included between them. If the smaller sphere increase in size till it ultimately coincides with the larger, find the final position of the centre of gravity of the

included mass. According to what law of density must matter be distributed over the surface of a sphere that its centre of gravity may coincide with the one just found?

Solution by (1) W. J. C. SHARP, M.A.; (2) *the* PROPOSER.

1. Let r, r be the radii of outer and inner spheres, and x the distance of centre of gravity, which evidently lies in the diameter through the point of contact, from this point; then

$$r'^3 r' + (r^3 - r'^3)\, x = r^3 r, \quad \text{therefore} \quad x = \frac{r^4 - r'^4}{r^3 - r'^3} = \tfrac{4}{3} r \text{ at limit.}$$

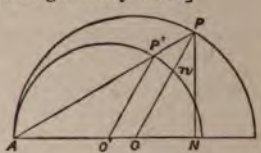
Now, if the masses of the rings be supposed collected at their centres, and $\phi(x)$ be the law of density, we have

$$\frac{\int_0^{2r} \phi(x)\, x\, dx}{\int_0^{2r} \phi(x)\, dx} = \tfrac{4}{3} r, \text{ and if } \phi(x) = \frac{d^2y}{dx^2} = \psi^2(x), \quad \frac{2r\psi'(2r) - \psi(2r) + \psi(0)}{\psi'(2r) - \psi'(0)} = \tfrac{4}{3} r;$$

hence, if $\psi(x) = y$, it satisfies the equation; $x \dfrac{dy}{dx} - 3y + 3c + 2c'x = 0$, and $y = c + c'x + c''x^3$, and $\phi(x) = \dfrac{d^2y}{dx^2} = 6c''x = \mu x$, which is the law of density of the collected masses, whilst that of the surface $= \frac{1}{2}\mu x r^{-1}$.

[This analytical solution has the advantage of being not a mere special result, but an example of a method that could be generally used.]

2. The density of the matter distributed on the surface of the sphere will follow the same law as that of the thickness between the two spheres when the inner is on the point of coinciding with the outer.



Let A be the point of contact; O′, O the centres of the inner and outer spheres; P any point on the outer; join AP, OP, meeting the inner sphere in P′, n respectively, and draw PN perpendicular to AO; then A is the centre of similitude, therefore P′P $\propto$ AP. Now, ultimately, Pn = PP′ cos APO; hence the required law of density $\propto$ AP cos PAO $\propto$ AN $\propto$ AP^2; it is therefore that of the square of the distance from a fixed point on the sphere.

6801. (By C. W. MERRIFIELD, F.R.S.)—Prove that the continued product $m^n!$, where m is prime, is divisible without remainder by m to the power of $(m^n - 1) : (m - 1)$, and the quotient is not again divisible by m.

Solution by W. B. GROVE, B.A.; BELLE EASTON; *and others.*

Consider first $m^3!$; the only factors divisible by m are

$m, 2m, 3m, \ldots m \cdot m, (m+1)\, m, \ldots 2m \cdot m, (2m+1)\, m, \ldots 3m^2, \ldots 4m^2, \ldots m^3$.

The number of these factors is m^2; dividing each of these by m, since m is prime, there remain only the quotients from the factors m^2, $2m^2$, ... m^3 still divisible by m; the number of these is m. Dividing these, there remains only one quotient, namely that from m^3, divisible still by m. Therefore $m^3!$ is divisible by m to the power (m^2+m+1) only, that is, $m^3-1 : m-1$. The proof may be easily completed by induction.

[The theorem results from a generalization of the following :—

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16	
1			1		1		1		1		1		1		1	2^3
			1				1				1				1	2^2
							1								1	2
															1	1

the units representing the number of times 2 enters. Summing these by lines, we have $2^3+2^2+2+1 = \frac{2^4-1}{2-1}$; and writing m for 2 and n for 4, which is obviously permissible, we get the theorem.]

6597. By Professor Townsend, M.A.)—A uniform flexible cord, in free equilibrium under the action of a central repulsive force varying inversely as the square of the distance, being supposed to have throughout its entire extent the tension to infinity under the action of the force; determine, for any assigned positions of its two terminal points with respect to the centre of force, the requisite length of the cord and the appropriate form of the catenary.

Solution by the Proposer; G. F. Walker, M.A.; *and others.*

The tension throughout the entire extent of a uniform flexible cord, in free equilibrium under the action of any central force, varying in all cases inversely as the perpendicular p upon the tangent from the centre of the force, and varying in the case in question inversely as the distances from the centre, in order to have its tension throughout that to infinity under the action of the force, the form of the catenary is consequently such that throughout its entire extent the ratio of p to r is constant. It is therefore an arc either of a circle or of a logarithmic spiral, according as its terminal radii are equal or unequal; and its length, in either case, being given immediately with the lengths and directions of its terminal radii, therefore &c.

6026. (By W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T.

Solution by W. J. C. SHARP, M.A.

Since the angles OQT and OTQ are supplementary, a circle can be described through OPTQN, the centre C of which is the intersection of PQ and TN. Then OPT is a constant angle, and so is OQP = OTP, and consequently TOP, hence the locus of T is an equiangular spiral; and since ONT = OQT a constant, OT : ON is constant; and since TON is a right angle, the locus of N is also an equiangular spiral, which is the evolute of the locus of T. (SALMON'S *Higher Plane Curves*, p. 281.)

6780. (By R. RAWSON.)—BOOLE has given (*Diff. Eqs.*, p. 459) the theorem from CURTIS (*Camb. Math. Jour.*, Vol. ix., p. 280)

$$\frac{d^2u}{dx^2}+2Q\frac{du}{dx}+\left\{Q^2+\frac{dQ}{dx}\pm c^2-\frac{m(m+1)}{x^2}\right\}u=0,$$

which can be integrated in finite terms when Q is any function of x. Show that the more general equation

$$\frac{d^2u}{dx^2}+2Q\frac{du}{dx}+\left\{Q^2+\frac{dQ}{dx}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}u=0$$

can be integrated in finite terms, where A is any constant, and $(2p+1)m=-4p$; p being any integer.

I. *Solution by* ARTHUR HILL CURTIS, LL.D., D.Sc.

The given equation is at once reducible to

$$\left\{\left(\frac{d}{dx}+Q\right)^2+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}u=0,$$

or, putting $\int Q\,dx=Q_1$, to

$$\left\{\frac{d^2}{dx^2}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}e^{Q_1}u=0;$$

and, putting $e^{Q_1}u=v$, we get

$$\left\{\frac{d^2}{dx^2}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}v=0,$$

or

$$\left\{x^2\frac{d^2}{dx^2}+Ax^{r+2}+\frac{(m-r)(m+r+4)}{4(m+2)^2}\right\}v=0.$$

Let $x=e^\theta$, and let $\frac{d}{d\theta}$ be denoted by D; then the equation is reducible to

$$\left\{D(D-1)+\frac{(m-r)(m+r+4)}{4(m+2)^2}+Ae^{(r+2)\theta}\right\}v=0,$$

or

$$\left\{\left(D-\frac{m-r}{2(m+2)}\right)\left(D-\frac{m+r+4}{2(m+2)}\right)+Ae^{(r+2)\theta}\right\}v=0,$$

or, putting $v=ze^{\frac{m-r}{2(m+2)}\theta}$,

$$\left\{D\left(D-\frac{r+2}{m+2}\right)+Ae^{(r+2)\theta}\right\}z=0.$$

Put $(r+2)\theta = (m+2)\phi$, then $D = \frac{d}{d\theta} = \frac{(r+2)}{(m+2)}\frac{d}{d\phi}$,

and the equation becomes

$$\left\{\left(\frac{r+2}{m+2}\right)^2 \frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right) + Ae^{(m+2)\phi}\right\} z = 0;$$

or, putting $e^\phi = y$, $\left\{y\frac{d}{dy}\left(y\frac{d}{dy}-1\right) + A\left(\frac{m+2}{r+2}\right)^2 y^{m+2}\right\} z = 0,$

or $$\left\{y^2\frac{d^2}{dy^2} + A\left(\frac{m+2}{r+2}\right)^2 y^{m+2}\right\} z = 0,$$

or $$\frac{d^2z}{dy^2} + A\left(\frac{m+2}{r+2}\right)^2 y^m z = 0,$$

and, if $m = -\frac{4p}{2p+1}$, this equation is identical with RICCATI's equation, which, as is well known, can be solved, when p is an integer, in a variety of ways. One method is given in my paper referred to by Mr. RAWSON.

When z has been found as a function of y, it is known as a function of x by the relation $y = e^\phi = (e^\theta)^{\frac{r+2}{m+2}} = x^{\frac{r+2}{m+2}}$, and u is known by the relation $u = e^{-Q_1}v = e^{-Q_1}x^{\frac{m-r}{2(m+2)}}z$.

II. *Solution by the* PROPOSER.

The relations $x_1 = x^n$, $y_1 = \frac{a_1 y}{anx^{n-1}} + \frac{n-1}{2anx^n}$ .....(1)

transform $\frac{dy_1}{dx_1} + ay_1^2 = bx_1^m + \frac{c}{x_1^2}$ into $\frac{dy}{dx} + a_1y^2 = Ax^r + \frac{c_1}{x^2}$.....(2, 3),

where $A = \frac{abn^2}{a_1}$, $c_1 = \frac{(4ac+1)n^2-1}{4a_1}$, $r = (m+2)n-2$.

If $c = 0$, and m satisfies RICCATI's condition of integrability, then equation (3) is soluble by substituting in (1) the value of y_1, obtained by RICCATI's process in (2).

Then $c_1 = \frac{n^2-1}{4a_1} = \frac{(r+2)^2-(m+2)^2}{4a_1(m+2)^2} = \frac{(r-m)(r+m+4)}{4a_1(m+2)^2}$;

and, when $a_1 = -1$, (3) becomes

$$\frac{dy}{dx} - y^2 = Ax^r + \frac{(m-r)(m+r+4)}{4(m+2)^2x^2} \quad \text{.....(4).}$$

RICCATI's condition is satisfied by $(2p+1)m + 4p = 0$, giving for the value of n the quantity $\frac{1}{2}(r+2)(2p+1)$.

Again, let $\frac{dy}{dx} + (Q-v)y = 0$, $\frac{du}{dx} + (Q+v)w = y$ .....(5, 6);

therefore $\frac{d}{dx}\left\{\frac{du}{dx} + (Q+v)w\right\} + (Q-v)\left\{\frac{du}{dx} + (Q+v)w\right\} = 0,$

or $$\frac{d^2u}{dx^2} + 2Q\frac{du}{dx} + \left\{Q^2 + \frac{dQ}{dx} + \frac{dv}{dx} - v^2\right\}u = 0 \quad \text{.....(7).}$$

If, then, v is determined from

$$\frac{dv}{dx} - v^2 = Ax^r + \frac{(m-r)(m+r+4)}{4(m+2)^2 x^2}$$

by means of (4), then the equation in the Question is soluble by means of (5) and (6). The values $r = 0$ and $A = \pm c^2$ give CURTIS's theorem.

6446. (By Professor GENESE, M.A.)—If $u \equiv (a, b, c, f, g, h)(x, y, 1)^2 = 0$ be the equation to a conic, prove that (1) the *directrices* are determined by the equation $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, where λ is the root of the equation $\lambda^2 - (a+b)\lambda + ab - h^2$; and (2) that, if α, β be the coordinates of the centre, the equation to the axes is

$$\frac{du}{dx} : \frac{du}{dy} = x - \alpha : y - \beta.$$

Solution by Professor WOLSTENHOLME, M.A.

If $u \equiv (a, b, c, f, g, h)(x, y, 1)^2$ be the equation of a conic referred to rectangular coordinates, the equation of its director circle is

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u.$$

Now the two directrices (real or impossible) of a conic pass through the four common points of a conic and its director circle, as is obvious by taking the forms $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x^2 + y^2 = a^2 + b^2$, which give $x^2 = \frac{a^4}{a^2 - b^2}$ taken simultaneously. Hence the equation of a pair of directrices will be of the form $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, λ being sufficiently determined by making the discriminant of this equation vanish. This gives the cubic

$$\begin{vmatrix} a^2 + h^2 - \lambda a, & h(a+b-\lambda), & ag + hf - \lambda g \\ h(a+b-\lambda), & b^2 + h^2 - \lambda b, & bf + hg - \lambda f \\ ag + hf - \lambda g, & bf + hg - \lambda f, & f^2 + g^2 - \lambda c \end{vmatrix} = 0;$$

but [as shown in the solution to Quest. 6038; *Reprint*, Vol. XXXII., p. 91], this equation reduces to $\Delta\lambda\,[(\lambda - a)(\lambda - b) - h^2] = 0$

[of which the significant factor might be obtained at once by making the equation parabolic].

The root $\lambda = 0$ gives us the two diameters through the four common points, viz., $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 0$; hence the equation of the two pairs of directrices is $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, where λ may have either value given by the equation $(\lambda - a)(\lambda - b) = h^2$, one root giving the

real directrices and the other the impossible. If $ab = h^2$ another root of the equation is zero, and we get the unique directrix $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u$, the terms of two dimensions disappearing. I arrived at the above result for the determinant in the investigation of this same equation. When I set the question in an examination here, I gave an example to illustrate the fact that, although this is the simplest form of general equation of directrices, it is by no means a convenient one to use in practice if the numerical coefficients are at all large. To get the *real* directrices, the greater algebraical root should be taken, if Δ, *i.e.*, the discriminant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, be negative. In oblique coordinates, the equations will be

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 - 2\frac{du}{dx}\frac{du}{dy}\cos\omega = 4\lambda u,$$

$$\lambda^2 - \lambda(a+b-2h\cos\omega) + (ab-h^2)\sin^2\omega = 0.$$

As an example (a very favourable one since there are no surds involved) take $u = 7x^2 + 23y^2 + 293 - 134y - 52x - 12xy$;

then $\Delta = 7 \times 23 \times 293 - 2 \times 67 \times 26 \times 6 - 7 \times 67^2 - 23 \times 26^2 - 293 \times 6^2$

$= 293 \times 125 - 67 \times (312 + 469) - 23 \times 676 = -37950$;

the roots of the equation $(\lambda - a)(\lambda - b) = h^2$ are 5 and 25; hence the equation of the real directrices is

$$(7x - 6y - 26)^2 + (-6x + 23y - 67)^2 = 25(7x^2 + 23y^2 + \ldots),$$

or $$90x^2 + 60xy + 10y^2 - 580(3x+y) + 2160 = 0,$$

or $(3x+y)^2 - 58(3x+y) + 216 = 0$, therefore $3x + y = 4$ or 54.

For the foci, $\frac{7x-6y-26}{3} = \frac{-6x+23y-67}{1} = \frac{26x+67y-293}{4 \text{ or } 54}$,

whence the major axis is $x - 3y + 7 = 0$; and we have, on substituting for x

$$5y - 25 = \frac{135y - 475}{4 \text{ or } 45},$$

(1) $4y - 20 = 29y - 95$, $y = 3$, and therefore $x = 2$.

(2) $54y - 270 = 29y - 95$, $y = 7$, and $x = 14$.

In reducing, all the coefficients need not be calculated (unless as a test of accuracy) since we know that the equation represents two *parallel* straight lines; it will be sufficient to find those of x^2, xy, x, and the constant term.

6722. (By D. Edwardes.)—Prove that the mean distance from its centre of all points within an oblate spheroid, of equatorial radius a, is

$$\tfrac{3}{8}(ae^{-1}\sin^{-1}e + b).$$

Solution by Rev. T. R. Terry, M.A., F.R.A.S.; Kate Gale; *and others.*

$$\text{Mean distance} = \frac{3}{4\pi a^2 b} 2 \iint dx\, dy\, 2\pi x\, (x^2+y^2)^{\frac{1}{2}} \text{ (over } \tfrac{1}{4} \text{ of ellipse)}$$

$$= \frac{1}{a^2 b} \int_0^b \left\{ \frac{a^3}{b^3} (b^2 - e^2 y^2)^{\frac{3}{2}} - y^3 \right\} dy = \frac{a}{e} \int_0^{\sin^{-1} e} \cos^4 \theta\, d\theta - \frac{b^3}{4a^3},$$

where $ey = b \sin\theta = \frac{3}{8}(ae^{-1} \sin^{-1} e + b)$.

6574. (By W. B. Grove, B.A.)—Suppose a series of 73 cards to be painted with red, blue, yellow, and green, every card but one receiving at least one colour. Let it be observed that 21 have *some* part coloured red, 48 blue, 31 yellow, and 46 green; also 14 have both red and blue upon them, 16 both red and green, 14 both blue and yellow, 28 both blue and green, 20 both yellow and green, and 9 have all four colours. Also 15 are painted with blue *alone*, 6 with yellow alone, but none with either red or green alone. Find the laws (designed or accidental) according to which the colours are arranged.

Solution by Hugh McColl, B.A.

Taking any card out of the series, and speaking of this card throughout, let r, b, y, g respectively denote the statements, it is *red*, it is *blue*, it is *yellow*, it is *green*; and let ϵ (an equivalent symbol for 1) denote the admitted statement, it is *one of the given series*. Also, let α_n denote the number of cards in the series to which any statement α is applicable, so that $\alpha_n = 73\alpha_\epsilon$, in which α_ϵ denotes the chance that α is true.

Then, our data are: $r_n = 21$, $b_n = 48$, $y_n = 31$, $g_n = 46$, $(r+b+y+g)_n = 72$, $(rb)_n = 14$, $(rg)_n = 16$, $(by)_n = 14$, $(bg)_n = 28$, $(yg)_n = 20$, $(rbyg)_n = 9$, $(r'by'g')_n = 15$, $(r'b'yg')_n = 6$, $(rb'y'g')_n = 0$, $(r'b'y'g)_n = 0$. Applying rules 2 and 4 of my fourth paper in the *Proceedings* of the London Mathematical Society, we readily get the 5 equations

$$(rby)_n + (rbg)_n + (ryg)_n + (byg)_n - (ry)_n = 27, \quad (rby)_n + (rbg)_n + (byg)_n = 32,$$

$$(rby)_n + (ryg)_n + (byg)_n - (ry)_n = 18, \quad (rby)_n + (ryg)_n + (rbg)_n - (ry)_n = 18,$$

$$(byg)_n + (ryg)_n + (rbg)_n = 27;$$

from which we get $(rby)_n = 14$, $(rbg)_n = 9$, $(ryg)_n = 9$, $(byg)_n = 9$, $(ry)_n = 14$. Applying rule 4 (when necessary) to determine the number of cards to which any four-factor statement (such as $rb'y'g$) is applicable, we get $(rbyg)_n = 9$, $(rbyg')_n = 5$, $(rb'yg)_n = 7$, $(r'byg)_n = 19$, $(r'by'g')_n = 15$, $(r'b'yg)_n = 11$, $(r'b'yg')_n = 6$, $(r'b'y'g')_n = 1$. The remaining 8 compound statements are not applicable to any card, and, reducing their sum to its primitive form, we get $r(b'y + by' + b'g') + r'(by + b'y'y) : 0$, which is equivalent to the equation $r = by + b'y'g$.

6667. (By the Editor.)—Given the distances of a point from three of the corners of a square, (1) construct the square; and prove (2) that if a, b, c be these distances and Δ the area of a triangle whose sides are a, $b\sqrt{2}$, c, the area of the square is $\frac{1}{2}(a^2+c^2) \pm 2\Delta$; also (3) extend the problem to the case of any rectangle or triangle given in species, showing that, in the case of a rectangle whose sides are as $m : n$ and area Σ, if Δ be the area of a triangle with sides a, b', c', where

$$b' = \frac{(m^2+n^2)^{\frac{1}{2}}}{n} b, \quad c' = \frac{m}{n} c, \quad \text{then } \Sigma = \frac{1}{m^2+n^2}\left\{mn(a^2+c^2) \pm 4n^2\Delta\right\},$$

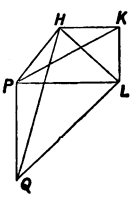
+ or − according as the angle subtended at the point by the diagonal is obtuse or acute.

I. *Solution by* G. Heppel, M.A.; Lizzie A. Kittudge; *and others.*

1. Let A, B, C be three consecutive corners of any square. Divide AB externally and internally in the ratio of $a : b$ by the points D, E. On DE draw a semicircle, which will be the locus of points whose distances from A and B are as $a : b$. Determine a similar locus for BC and $b : c$. These circles will, in general, cut in two points. If Q be either of these, at the given point P make angles HPK, KPL equal to AQB, BQC, and take PH = a, PK = b, PL = c; HKL will be the required square.

2. The construction for a rectangle or triangle given in species is precisely similar, starting with any rectangle or triangle of the given shape.

3. Draw PQ at right angles and equal to PL. Join HL, HQ, LQ. Then $\angle$ KLP = HLQ, and HL = KL$\sqrt{2}$, QL = PL$\sqrt{2}$; hence HQ = PK$\sqrt{2}$. Now



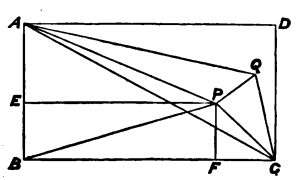
$$HL^2 = \text{twice } \square = a^2+c^2-2ac\cos HPL$$
$$= a^2+c^2-2ac\sin HPQ = a^2+c^2-4\Delta HPQ,$$

therefore $\square = \frac{1}{2}(a^2+c^2)-2\Delta$.

In the case where the angle HPL is obtuse, we shall find that $\square = \frac{1}{2}(a^2+c^2)+2\Delta$.

II. *Solution by the* Proposer.

1. Taking the general case of a right-angled triangle ABC and rectangle ABCD (the case for *any* triangle being precisely similar), draw PE, PF parallel to the sides, and put



$$PA = a, \ PB = b, \ PC = c,$$
$$AB = mx, \ BC = nx;$$

then we have

$$BE = \frac{b^2+m^2x^2-a^2}{2mx}, \quad BF = \frac{b^2+n^2x^2-c^2}{2nx}; \text{ and } BE^2+BF^2 = BP^2;$$

whence $n^2\left\{m^2x^2+(b^2-a^2)\right\}^2 + m^2\left\{n^2x^2+(b^2-c^2)\right\}^2 = 4m^2n^2b^2$;

$$m^2n^2(m^2+n^2)x^4 - 2m^2n^2(a^2+c^2)x^2 + \left\{m^2(b^2-c^2)^2 + n^2(b^2-a^2)^2\right\} = 0;$$

$$\frac{m^2+n^2}{a^2+c^2}x^2 = 1 \pm \left\{1 - \frac{(1+m^2n^{-2})(b^2-c^2)^2 + (1+n^2m^{-2})(b^2-a^2)^2}{(a^2+c^2)^2}\right\}^{\frac{1}{2}};$$

$$(m^2+n^2)\,x^2 = a^2+c^2 \pm \left\{ (a^2+c^2)^2 - \frac{m^2+n^2}{n^2}(b^2-c^2)^2 - \frac{m^2+n^2}{m^2}(b^2-a^2)^2 \right\}^{\frac{1}{2}}$$

$$= a^2+c^2 \pm \frac{n}{m}\left\{ 2\,(b'^2c'^2+c'^2a^2+a^2b'^2) - (a^4+b'^4+c'^4) \right\}^{\frac{1}{2}}$$

$$= a^2+c^2 \pm \frac{n}{m}\,.\,4\Delta\,;$$

therefore $\Sigma = mnx^2 = \dfrac{1}{m^2+n^2}\left\{ mn\,(a^2+c^2) \pm 4n^2\Delta \right\}$.

2. *Otherwise:* draw PQ perpendicular to PC, and make $\angle$ QCP = ACB; then, in the triangles ACQ, BCP, we have

$$AC = (1+m^2n^{-2})^{\frac{1}{2}}\,BC,\quad CQ = (1+m^2n^{-2})^{\frac{1}{2}}\,CP,\quad \angle ACQ = BCP,$$

therefore $AQ = (1+m^2n^{-2})^{\frac{1}{2}}\,b = b'$; also $PQ = c'$;

hence $AC^2 = AP^2 + PC^2 - 2AP\,.\,PC\cos APC$

$$= a^2+c^2+2AP\,.\,\frac{n}{m}\,PQ\sin APQ = a^2+c^2+\frac{n}{m}\,.\,4\Delta\,;$$

but $AC^2 = (m^2+n^2)\,x^2$, and $\Sigma = mnx^2 = \dfrac{mn}{m^2+n^2}\,.\,AC^2$;

therefore $\Sigma = \dfrac{mn}{m^2+n^2}(a^2+c^2) + \dfrac{4n^2}{m^2+n^2}\Delta = \dfrac{1}{m^2+n^2}\left\{ mn\,(a^2+c^2) + 4n^2\Delta \right\}$.

3. In the case of a square, we have $m = n$, and then $\Sigma = \frac{1}{2}(a^2+c^2) \pm 2\Delta$.

6612. (By E. B. Elliott, M.A.)—A spherical surface of radius a, made of thin perfectly flexible material whose internal side is reflecting, is deformed without either crumpling or stretching, and assumes the form of a surface of revolution. From a point Q on the axis of this surface, and within it, a small pencil of light is incident on it at any point A. Show that, if q_1, q_2 be the primary and secondary foci of the reflected pencil, and if R be the image of Q by reflection in the tangent plane at A,

$$Rq_1\,.\,Rq_2 : Aq_1\,.\,Aq_2 = 4QA^2 : a^2.$$

Solution by G. F. Walker, M.A.; Lizzie A. Kittudge; *and others.*

We have $\dfrac{1}{v_1} + \dfrac{1}{u} = \dfrac{2}{\rho_1\cos\phi}$,

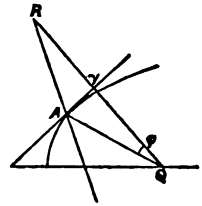
and $\dfrac{1}{v_2} + \dfrac{1}{u} = \dfrac{2\cos\phi}{\rho_2}$,

or $\dfrac{Rq_1}{Aq_1\,.\,AQ} = \dfrac{2}{\rho_1\cos\phi}$, $\dfrac{Rq_2}{Aq_2\,.\,AQ} = \dfrac{2\cos\phi}{\rho_2}$,

therefore $\dfrac{Rq_1\,.\,Rq_2}{Aq_1\,.\,Aq_2\,.\,AQ^2} = \dfrac{4}{\rho_1\rho_2} = \dfrac{4}{a^2}$

by the given condition. Therefore

$$Rq_1\,.\,Rq_2 : Aq_1\,.\,Aq_2 = 4AQ^2 : a^2.$$



6257. (By A. Buchheim, Ph.D.)—Show that

$$\iiint \frac{\sum_1^3 x_i^6 + 3\sum_1^3\sum_1^3 x_i^2 x_j^4 + 2x_1^2x_2^2x_3^2}{(\sum_1^3 x_i^2)^6}\, dx_1 dx_2 dx_3 = \frac{4\pi}{3abc};$$

the integration extending over the whole space outside the pedal of the ellipsoid of semiaxes a, b, c with regard to its centre the origin of co-ordinates.

Solution by the Proposer; Professor Evans, M.A.; *and others.*

The coefficient of $dx_1 dx_2 dx_3$ is the Jacobian of y_1, y_2, y_3 with regard to x_1, x_2, x_3, the points x, y being inverse points with regard to the sphere of radius unity described about the origin; hence the integral gives the volume of the inverse of the pedal of the ellipsoid, that is, of the ellipsoid $a^2x_1^2 + b^2x_2^2 + c_2x_3^2 = 1$; but this is $\frac{4\pi}{3abc}$.

6529. (By E. W. Symons, M.A.)—From a point four normals are drawn to a conic; prove that, if the line joining two of their feet move parallel to itself, the line joining the other two will move parallel to itself; and find the locus of the point in order that this may be possible.

Solution by G. F. Walker, M.A.; Christine Ladd; *and others.*

The normals at the ends of the chords (1) meet in the point (2),

$$\frac{lx}{a} + \frac{my}{b} = 1,\ \frac{x}{la} + \frac{y}{mb} + 1 = 0 \ldots(1),\ \frac{a^2-b^2}{l^2+m} = \frac{by}{m(l^2-1)} = \frac{ax}{-l(m^2-1)} \ldots(2),$$

and if one be fixed in direction, so is the other.

If we take $l \sec\alpha = m \operatorname{cosec}\alpha = \lambda$, so that the chord is parallel to the tangent at the point α, we get, to find (x, y),

$$\frac{a^2-b^2}{\lambda} = \frac{by}{\sin\alpha(\lambda^2\cos^2\alpha - 1)} = \frac{ax}{-\cos\alpha(\lambda^2\sin^2\alpha - 1)} = \frac{ax\cos\alpha + by\sin\alpha}{\cos 2\alpha}$$

$$= \frac{ax\sec\alpha + by\operatorname{cosec}\alpha}{\lambda^2\cos 2\alpha}, \text{ and the locus of } (x, y) \text{ is}$$

$$(ax\cos\alpha + by\sin\alpha)\left(\frac{ax}{\cos\alpha} + \frac{by}{\sin\alpha}\right) = (a^2-b^2)^2\cos^2 2\alpha.$$

6431. (By E. W. Symons, M.A.)—Prove that the locus of a point, the normals from which to a given ellipse form a harmonic pencil, is

$$(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2 \cdot c^4x^2y^2 = 0 \quad (c^2 \equiv a^2 - b^2).$$

Solution by the PROPOSER; C. H. SWIFT, B.A.; *and others.*

The equation of a normal may be written

$$y = mx - mc^2(a^2 + m^2b^2),$$

whence $m^4b^2x^2 - 2m^3b^2xy + m^2(a^2x^2 + b^2y^2 - c^4) - 2a^2mxy + a^2y^2 = 0$

is the relation connecting the angles of inclination to the x-axis of the four normals from (x, y). Now the condition for an harmonic pencil is the invariant T = 0 (see SALMON'S *Higher Algebra*, p. 187, art. 206). If $\Sigma \equiv a^2x^2 + b^2y^2 - c^4$, this gives, for locus of point,

$$\tfrac{1}{6}a^2b^2x^2y^2\Sigma + \tfrac{1}{12}a^2b^2x^2y^2\Sigma - \tfrac{1}{4}a^4b^2x^4y^2 - \tfrac{1}{4}a^2b^4x^2y^4 - \tfrac{1}{216}\Sigma^3 = 0,$$

$$a^2b^2x^2y^2\left\{\tfrac{1}{6}\Sigma + \tfrac{1}{12}\Sigma - \tfrac{1}{4}\Sigma - \tfrac{1}{4}c^4\right\} - \tfrac{1}{216}\Sigma^3 = 0,$$

or $$54a^2b^2c^4x^2y^2 + \Sigma^3 = 0.$$

6555. (By H. STEWART, B.A.)—If

$$\frac{bc(y+z) - a^2x}{a(b^2 + c^2 - bc)} = \frac{ca(z+x) - b^2y}{b(c^2 + a^2 - ca)} = \frac{ab(x+y) - c^2z}{c(a^2 + b^2 - ab)},$$

prove that $(a^2x + b^2y + c^2z)(b^2c^2 + c^2a^2 + a^2b^2) = 3a^2b^2c^2(x + y + z)$.

Solution by the Rev. J. L. KITCHIN, M.A.; J. O'REGAN; *and others.*

Putting each of the fractions $= \lambda$, and solving, we easily obtain

$$x = \lambda\frac{bc}{a}, \quad y = \lambda\frac{ac}{b}, \quad z = \lambda\frac{ab}{c},$$

therefore $a^2x + b^2y + c^2z = 3\lambda abc$; also $x + y + z = \lambda\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$,

hence $abc(x + y + z) = \lambda(b^2c^2 + a^2c^2 + a^2b^2)$, therefore &c.

6465. (By E. W. SYMONS, M.A.)—Prove that the line joining the point where $ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$ is cut by $a'x^2 + 2h'xy + b'y^2 = 0$ is

$$\frac{(\lambda a' - a)}{g}x + \frac{(\lambda b' - b)}{f}y = 2, \text{ wherein } \lambda \equiv \frac{af^2 + bg^2 - 2fgh}{a'f^2 + b'g^2 - 2fgh'}.$$

Solution by R. E. RILEY, M.A.; D. EDWARDES; *and others.*

Assume the required equation to be $lx + my = 1$, then

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my) = 0$$

is identic with $a'x^2 + 2h'xy + b'y^2 = 0$. Hence we get two linear equations in l, m, which give the result stated in the Question.

6564. (By Professor TOWNSEND, F.R.S.)—A point, taken arbitrarily in the plane of a uniform circular lamina attracting inversely as the square of the distance, and its reflexion with respect to the centre, being supposed the foci, if internal of an ellipse, and if external of an hyperbola, having the containing diameter for transverse axis of figure; show that the potential of the attraction for either point is equal to the mass per unit of area of the lamina, multiplied into the circumference of the ellipse in the former case, and into the difference between the circumference of the hyperbola and the sum of its asymptotes in the latter case.

Solution by the PROPOSER.

Denoting by a the radius of the lamina, by m its mass per unit of area, by c the distance of the point from its centre, by l and θ the length and inclination to the containing diameter of any chord passing through the point, and by V the potential of the attraction, distinguished as V_i or V_e according as the point is internal or external to the mass of the lamina; then, since manifestly, in either case, $dV = mld\theta$, and $l = 2(a^2 - c^2\sin^2\theta)^{\frac{1}{2}}$, therefore, at once, in the former case $V_i = 4m\int_0^{\frac{1}{2}\pi}(a^2 - c^2\sin^2\theta)^{\frac{1}{2}}d\theta$, and in the latter case $V_e = 4m\int_0^{\sin^{-1}\frac{a}{c}}(a^2 - c^2\sin^2\theta)^{\frac{1}{2}}d\theta$; and therefore, &c.

6395. (By E. W. SYMONS, M.A.)—Three normals being drawn to an ellipse from a point in its evolute; prove (1) that the locus of the centre of the circle passing through their feet is $4(a^2x^2 + b^2y^2)^3 = a^2b^2(a^2 - b^2)^2x^2y^2$; and (2) that one of the common chords of the ellipse and circle passes through the centre of the ellipse.

Solution by G. F. WALKER, M.A.; *the* PROPOSER; *and others.*

It is not hard to show that the normals at the points where

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0 \text{ and } \frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} + 1 = 0$$

meet the ellipse, pass through a point on the evolute.

Thus $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda\left(\frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} + 1\right)\alpha$$

must be a circle passing through $(a\cos\phi, b\sin\phi)$, therefore

$$\alpha \equiv bx\sin\phi - ay\cos\phi;$$

and this being one of the common chords of ellipse and circle, and passing through the centre, proves (2).

Again, expressing that coefficient of x^2 = that of y^2, we get

$$\lambda \equiv \frac{(a^2 - b^2)\sin\phi\cos\phi}{ab(a^2\cos^2\phi + b^2\sin^2\phi)}, \text{ and the equation of the circle is}$$

$$x^2+y^2+\frac{(a^2-b^2)}{a}\sin^2\phi\cos\phi\,x+\frac{(b^2-a^2)}{b}\sin\phi\cos^2\phi\,y-a^2\cos^2\phi-b^2\sin^2\phi=0;$$

therefore, if (x, y) be the centre, we have

$$2ax=-(a^2-b^2)\sin^2\phi\cos\phi,\quad 2by=(a^2-b^2)\sin\phi\cos^2\phi;$$

and therefore, eliminating ϕ, we get the required result.

6811. (By the Rev. C. Taylor, D.D.)—Focal chords of a parabola at right angles to one another meet the directrix in T, t. Show that (1) the bisectors of the angles between the tangents from either of the points T, t are parallel to the tangents from the other; and (2) every pair of the four tangents intersect at constant angles.

Solution by F. Budd, M.A.; Kate Gale; *and others.*

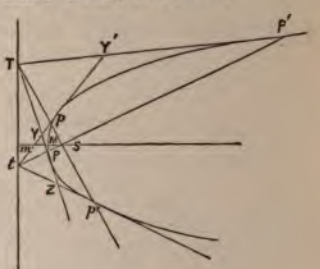
Since the tangents at P and P' intersect at T at right angles,

$$\angle Y'TS = SPT$$
$$= \angle SMP = \angle YMm,$$

and $\angle Y'pT = \angle Spm = \angle Smp$;

$$\therefore\ \angle TY'Y = \angle mYM = \angle TYY'$$
$$= \tfrac{1}{4}\pi = PZt, \text{ \&c.}$$

Therefore the bisector of $\angle PTP'$ is at right angles to YY', and therefore parallel to tp', &c.



An Analysis of Relationship.

By Alexander Macfarlane, M.A., D.Sc., F.R.S.E.

In this article I propose to give an abstract of several papers on an Analysis of Relationship, which I have recently contributed to the Royal Society of Edinburgh (Proceedings for May 1879, Dec. 1880, and March 1881). The method is a development in a special direction of the principles of the *Algebra of Logic*, and as finally developed enables one to solve problems about relationships by purely analytical processes.

1. The subject considered is, in its widest extent, mankind; in a particular investigation, it may be any specified portion of that natural class. Let U denote any man in the community of men considered, U_A the man whose name is A, U_B the man whose name is B,—the latter may be written A, B, the U being left to be understood as being constant throughout the given investigation.

2. Let cA denote a child of A; then c^{-1}A will denote a parent of A, ccA a grandchild of A, cc^{-1}A a child of a parent of A, $c^{-1}c$A a parent of a child of A, $c^{-1}c^{-1}$A a grandparent of A. These expressions can be denoted without ambiguity by c^2A, c^{1-1}A, c^{-1+1}A, c^{-2}A respectively. The index in c^{1-1} may or may not reduce to 0; if it is irreducible, the relationship means *brother or sister*; if it reduces, the meaning is *self*.

3. The symbol Σ is used to denote *all*, as ΣcA all the children of A. If the value of "the all" is specified, then Σ is replaced by a number with a dot over it, as $\dot{3}c$A, the three children of A. Numbers are also required to express a *definite some*, as $2c$A two children of A. The former may be called *complete*, the latter *partial* numbers. Numbers are also required in their ordinary sense of coefficients, in which case they may be distinguished by having a larger size as $2\Sigma c$A twice the children of A. Subscript numbers as in c_1A, c_2A express the eldest child of A, the second child of A, etc.

4. In such an expression as $(3+2)c$A, where two partial numbers are connected by +, it is not necessary that each of the individuals counted in the $3c$A should be different from each of the individuals counted in the $2c$A. The expression $(3+2)c$A is equivalent either to $5c$A or $(3+2.1)c$A (where the dot marks off the coefficient) or $(1+2.2)c$A, where the terms connected are now exclusive of one another. Similarly $(3-2)c$A is not necessarily equivalent to $1c$A. It may be irreducible, or it may reduce to $(2-1)c$A, or it may reduce further to cA. Such an expression as $(\dot{3}+2)c$A is equivalent to $(1+2\cdot2)c$A, and $(\dot{3}-2)c$A to cA.

5. The symbol Σ placed before a sum of terms is distributive; a definite or partial number is non-distributive.

6. Let the order of a relationship refer to the number of times c enters into its expression, whether directly or inversely; then the general relationships of the n^{th} order are obtained by expanding $(c+c^{-1})^n$, provided the consecutiveness of the symbols in the products be preserved.

For example,

Term.	General Meaning.	Meaning, if irreducible.
c^3	Great-grandchild	
c^{2-1}	Grandchild of parent	Nephew or niece
c^{1-1+1}	Child of parent of child	Step-child
c^{1-2}	Child of grandparent	Uncle or aunt
c^{-1+2}	Parent of grandchild	Son-in-law or Daughter-in-law
c^{-1+1-1}	Parent of child of parent	Step-parent
c^{-2+1}	Grandparent of child	Father-in-law or Mother-in-law
c^{-3}	Great-grandparent	

7. *Laws of the Index.* If two consecutive numbers in an index have the same sign, they can be added together; but, if they have different signs, they cannot be simplified by ordinary subtraction. Suppose the index is $p-q$, it may be irreducible, or it may reduce to $(p-1)-(q-1)$, or further to $(p-2)-(q-2)$, and so on until one of the indices is reduced to 0.

8. The general relationships are rendered more specific by introducing before a c or c^{-1} one of the sex-symbols m and f, the former denoting *male*,

the latter *female*. Thus mc denotes son, fc daughter, mc^{-1} father, fc^{-1} mother, $mcmc^{-1}$ son of the father, $mcfc^{-1}$ son of the mother.

9. A general relationship may be expanded into specific relationships by putting in $m+f$ before one or more of the symbols.

Thus
$$\Sigma c^2\mathrm{A} = \Sigma c\,(m+f)\,c\mathrm{A} = \Sigma\,(cmc+cfc)\,\mathrm{A}$$
$$= \Sigma\,(m+f)\,c^2\mathrm{A} = \Sigma\,(mc^2+fc^2)\,\mathrm{A}$$
$$= \Sigma\,(m+f)\,c\,(m+f)\,c\mathrm{A} = \Sigma\,(mcmc+mcfc+fcmc+fcfc)\,\mathrm{A}.$$

10. *Laws of Reduction.*—If the sex-symbols preceding and succeeding the expression c^{1-1} are the same, the expression may reduce to 1; but if the sex-symbols are different, the expression cannot reduce to 1. If the sex-symbols preceding and succeeding the expression c^{-1+1} are the same, the expression must reduce to 1; and if these are different, the expression cannot reduce to 1.

11. Name-symbols may be inserted before a relationship-symbol, as $c\mathrm{A}c^{-1}\mathrm{B}$, denoting a child of A who (that is, A) is a parent of B. When m or f is also specified, the proper position for the name-symbol is before the sex-symbol, as $c\mathrm{A}mc^{-1}\mathrm{B}f$, a child of A who is the father of B a woman.

12. A compound relationship due to the coexistence of several elementary relationships may be denoted by writing them after one another and using a dot to separate, as $cmc^{-1}\mathrm{A}\,.\,cfc^{-1}\mathrm{B}$, a child of the father of A, and also a child of the mother of B. If the terminal names be the same for all the elementary relationships, only the last need be written.

13. The Σ of a compound relationship is conditioned by the Σ's of the components. We may write $\Sigma\,(cm\mathrm{A}\,.\,cf\mathrm{B}) = \Sigma cm\mathrm{A}\,.\,\Sigma cf\mathrm{B}$. Here the Σ of the resultant term cannot be greater than the Σ of either component; it cannot be less than their sum minus the Σ of $\Sigma\mathrm{U}$; and it cannot be less than 0.

14. A compound relationship whose elementary relationships are specific forms of the same general relationship may be denoted by the expression for the general relationship with a vinculum over it and an index expressing the number of times the general relationship occurs. For example, $c^{\overline{1-1}^2}\mathrm{A} = cmc^{-1}\,.\,cfc^{-1}\mathrm{A}$. Thus $c^{\overline{1-1}^2}$ denotes a full brother or sister of A. Similarly $c^{\overline{1-1}^1}\mathrm{A}$ denotes a half brother or sister of A; and this notation gives us $c^{\overline{1-1}^0}\mathrm{A}$ as the expression for a non-brother or non-sister of A. This index does not coincide with the Boolian index; for the latter denotes the combination of the same relationship with itself.

15. Let r denote any general relationship, and suppose that it can assume n principal specific forms; then
$$\Sigma r\mathrm{A} = \Sigma\bar{r}^1 + 2\Sigma\bar{r}^2 + 3\Sigma\bar{r}^3 + \ldots + n\Sigma\bar{r}^n.$$
Several of these terms may be non-existent by reason of the Laws of Marriages.

16. *Transformation of a Conditional Equation.* — By a conditional equation is meant one which is true of a particular man as A or B, but not of any man U. For example, mc^{-1} Dick = mc Tom. Rule.—*Any symbol may be taken from the front of one side of an equation, provided its reciprocal be placed in front of the other side.* The reciprocal of m is m, that of f is f. Also $mm = m$, $ff = f$, $mf = 0$. For example, in the case of the above equation, Dick = cmc Tom; Tom = $c^{-1}mc^{-1}$ Dick. The reciprocal of Σ denoted by $\frac{1}{\Sigma}$ means *each of all*; $\frac{1}{3}$ means *each of three*.

17. *Transformation of a Universal Equation.*—By a Universal Equation

is meant one which is true for any person whatever, or for any person living under the laws of a given community. An example of the former is $\Sigma c \,.\, c^{-1}U = 0$; of the latter $\Sigma em \,.\, cfcmU = 0$.

RULE.—*A universal equation may be transformed in accordance with the previous Rule, not only at the front of the factors, but also at their end.*

6653. (By W. R. WESTROPP ROBERTS, M.A.) — A heavy elliptical lamina is placed in contact with a vertical wall OA, and with a horizontal wall OB, and moves in a vertical plane. Show that, the walls being smooth, the lamina leaves the vertical wall when

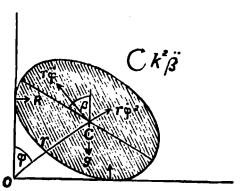
$$2(\sin\alpha - \sin\phi)\left\{1 + \frac{r^4a^2b^2\sin^2\phi\cos 2\phi}{(r^4\sin^2\phi\cos^2\phi - a^2b^2)(5r^4\sin^2\phi\cos\phi - 4a^2b^2)}\right\} = \sin\phi,$$

where $r^2 = a^2 + b^2$, $\phi = \widehat{COB}$, C being the centre of the lamina.

Solution by D. EDWARDES; CHARLOTTE A. SCOTT; *and others.*

Measuring ϕ so that $\frac{d\phi}{dt}$ is positive, and taking the mass of the body to be unity, the equation of energy is

$$\dot\phi^2 + \tfrac{1}{4}\left(\frac{d\beta}{dt}\right)^2 = \frac{2g}{r}(\cos\alpha - \cos\phi),$$



β being the inclination of major axis to the vertical at time t. But, by geometry, we have

$$\left(\frac{d\beta}{dt}\right)^2 = \frac{r^4\sin^2\phi\cos^2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\left(\frac{d\phi}{dt}\right)^2;$$

whence the equation above becomes

$$\dot\phi^2\left\{1 + \tfrac{1}{4}\cdot\frac{r^4\sin^2\phi\cos^2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right\} = \frac{2g}{r}(\cos\alpha - \cos\phi).$$

Reversing the effective forces, as in the figure, and resolving horizontally, we have

$$R + r\dot\phi^2\sin\phi = r\ddot\phi\cos\phi;$$

therefore R vanishes when $\dot\phi^2\sin\phi = \ddot\phi\cos\phi$, that is, when

$$\dot\phi^2\left\{2\tan\phi\left[1 + \frac{r\sin^2\phi\cos^2\phi}{4(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right] + \frac{r^4}{16}\frac{d}{d\phi}\left[\frac{\sin^2 2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right]\right\} = \frac{2g}{r}\ \text{in}\ \phi,$$

or, performing the differentiation and substituting for $\dot\phi^2$ its value, when

$$2(\cos\alpha - \cos\phi)\left\{1 - \frac{r^4a^2b^2\cos^2\phi\cos 2\phi}{(r^4\sin^2\phi\cos\phi - a^2b^2)(5r^4\sin^2\phi\cos^2\phi - 4a^2b^2)}\right\} = \cos\phi,$$

which is equivalent to the PROPOSER's result.

6013. (By Professor MINCHIN, M.A.)—If at any point in a body subject to stress the principal stresses consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $\sqrt{AC}$, and find the plane on which it is exerted. If the principal stresses are a tension of intensity A and two pressures of intensities B and C (B > C), show that the maximum intensity of shearing stress is $\sqrt{AB}$, and find the plane on which it is exerted.

Solution by W. H. BESANT, M.A., F.R.S.

The stress-quadric is $Ax^2 + By^2 - Cz^2 = f$, and the plane $lx + my + nz = 0$, with the condition $Al^2 + Bm^2 - Cn^2 = 0$, is a plane of shearing stress.

The tangent plane at (lr, mr, nr) is $Alrx + Bmry - Cnrz = f$. The perpendicular $p = f \div (A^2l^2 + B^2m^2 + C^2n^2)^{\frac{1}{2}}$, therefore the shearing stress $F = \frac{f}{pr} = (A^2l^2 + B^2m^2 + C^2n^2)^{\frac{1}{2}}$. We hence obtain, taking $l^2 + m^2 + n^2 = 1$, $F^2 = AC - m^2(B + C)(A - B)$, which is a maximum when $m = 0$; and then we have $l^2 = \frac{C}{A + C}$, and $n^2 = \frac{A}{A + C}$.

[Another solution is given in *Reprint*, Vol. XXXIII., pp. 80, 81.]

6700. (By Professor MOREL.)—On considère un cercle, un triangle inscrit ABC, et un triangle circonscrit A′B′C′, tel que les points de contact du second sont aux sommets du premier. D'un point quelconque M de la circonférence, on abaisse des perpendiculaires MP, MQ, MR sur les côtés du premier triangle et des perpendiculaires MP′, MQ′, MR′ sur les côtés du second. Démontrer que l'on a toujours

$$MP \cdot MQ \cdot MR = MP' \cdot MQ' \cdot MR'.$$

Solution by the Rev. T. R. TERRY, F.R.A.S.; G. HEPPEL, M.A.; *and others.*

Taking ABC as the triangle of reference, the equations to the circle and the side B′C′ are $a\beta\gamma + b\gamma + a\beta = 0$, $b\gamma + c\beta = 0$.....(1, 2); hence, if $(\alpha', \beta', \gamma')$ be the coordinates of M, a point on the circle, we have

$$\text{length } MP' = \pm \frac{b\gamma' + c\beta'}{(b^2 + c^2 - 2bc\cos A)^{\frac{1}{2}}} = \frac{\beta'\gamma'}{\alpha'}, \text{ by (1);}$$

therefore $MP' \cdot MQ' \cdot MR' = \alpha'\beta'\gamma' = MP \cdot MQ \cdot MR.$

[If O be the centre of the circle, $\frac{MQ'}{MB} = \frac{MB}{2OB}$, $\frac{MP}{MB} = \frac{MC}{2OC}$;

therefore $\frac{MP}{MQ'} = \frac{MC}{MB}$; similarly $\frac{MQ}{MR'} = \frac{MA}{MC}$, $\frac{MR}{MP'} = \frac{MB}{MA}$;

therefore $MP \cdot MQ \cdot MR = MP' \cdot MQ' \cdot MR'.$]

6040. (By Professor LLOYD TANNER, M.A.)—If

$$b^{n/a} \equiv b\,(b-a)\,(b-2a)\,\ldots\,[b-(n-1)\,.\,a],$$

prove that
$$\begin{vmatrix} b^{1/b}, & 0, & 0, & \ldots, & b^{n/a} \\ (2b)^{1/b}, & (2b)^{2/b}, & 0, & \ldots, & (2b)^{n/a} \\ (3b)^{1/b}, & (3b)^{2/b}, & (3b)^{3/b}, & \ldots, & (3b)^{n/a} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (nb)^{1/b} & (nb)^{n/b}, & (nb)^{3/b}, & \ldots, & (nb)^{n/a}-(nb)^{n/b} \end{vmatrix} = 0.$$

Solution by the PROPOSER.

We may write

$$c^{n/a} = p_0 + p_1 c^{1/b} + p_2 c^{2/b} + \ldots + p_{n-1} c^{n-1/b} + p_n c^{n/b},$$

the coefficients p being independent of c. For each side is a polynomial of the n^{th} degree in c, and p_n, p_{n-1}, &c. can be determined in succession by comparing coefficients of c^n, c^{n-1}, &c. on each side. In particular $p_n = 1$; also $p_0 = 0$, since c is a factor of every other term. Thus

$$p_1 c^{1/b} + p_2 c^{2/b} + \ldots + p_{n-1} c^{n-1/b} - (c^{n/a} - c^{n/b}) = 0.$$

If we form n equations from this by giving c, n different values, we can eliminate p_1, p_2, &c. The relation given in the question is obtained by putting b, $2b$, $3b$, &c. in turn for c, noticing that $(rb)^{s/b} = 0$ when $s > r$.

6175. (By the Rev. W. ROBERTS, M.A.)—P, Q, R are three points on an equilateral hyperbola whose centre is O, such that OQ bisects the angle POR; M is the middle point of the chord PQ, and N the middle point of the chord QR; express the ratio $\frac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}}$ in terms of the ratio $\frac{\text{OP}}{\text{OR}}$.

Solution by W. J. C. SHARP, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let $x^2-y^2 = a^2$ be the equation to the hyperbola, and θ, $\frac{1}{2}(\theta+\phi)$, ϕ the angles made by OP, OQ, OR respectively with the axis; then we have

$$\text{OP}^2 = a^2 \sec 2\theta, \quad \text{OQ}^2 = a^2 \sec(\theta+\phi), \quad \text{OR}^2 = a^2 \sec 2\phi,$$

$$\text{PQ}^2 = \text{OP}^2 + \text{OQ}^2 - 2\,.\,\text{OP}\,.\,\text{OQ}\cos\tfrac{1}{2}(\theta-\phi),$$

$$4\,.\,\text{OM}^2 = \text{OP}^2 + \text{OQ}^2 + 2\,.\,\text{OP}\,.\,\text{OQ}\cos\tfrac{1}{2}(\theta-\phi);$$

therefore $4\,.\,\text{OM}^2\,.\,\text{PQ}^2 = \text{OP}^4 + \text{OQ}^4 - 2\,.\,\text{OP}^2\,.\,\text{OQ}^2\cos(\theta-\phi)$;

similarly $4\,.\,\text{ON}^2\,.\,\text{QR}^2 = \text{OQ}^4 + \text{OR}^4 - 2\,.\,\text{OQ}^2\,.\,\text{OR}^2\cos(\theta-\phi)$;

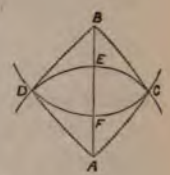
$$\therefore \left(\frac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}}\right)^2 = \frac{\sec^2 2\theta + \sec^2(\theta+\phi) - 2\sec 2\theta \sec(\theta+\phi)\cos(\theta-\phi)}{\sec^2 2\phi + \sec^2(\theta+\phi) - 2\sec 2\phi \sec(\theta+\phi)\cos(\theta-\phi)}$$

$$= \frac{\sec^2 2\theta}{\sec^2 2\phi} \cdot \frac{\cos^2(\theta+\phi) + \cos^2 2\theta - 2\cos 2\theta \cos(\theta+\phi)\cos(\theta-\phi)}{\cos^2(\theta+\phi) + \cos^2 2\phi - 2\cos 2\phi \cos(\theta+\phi)\cos(\theta-\phi)}$$

$$= \frac{\sec^2 2\theta}{\sec^2 2\phi} = \frac{\text{OP}^4}{\text{OR}^4}; \quad \text{therefore} \quad \frac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}} = \frac{\text{OP}^2}{\text{OR}^2}.$$

6193. (By Professor Seitz, M.A.)—Two equal small circles are drawn so as to intersect on the surface of a sphere of radius r; show that the average area of the spheric surface common to the two equal segments cut from the sphere is $(3\pi-8)\,r^2$.

Solution by the Proposer.



Let A and B be the poles of the two circles, C and D their intersections. Join AC, AD, BC, BD, and AB by arcs of great circles. Now, giving AC or BC, and AB all the values within the limits of the question, we are required to find the average area of the surface CEDF; and we may evidently consider one of the poles, as A, to be fixed. Let arc $AC=BC=rx$, $AB=ry$, $\angle CAB=\theta$, $\angle ACB=2\phi$; then we have

$$\tan\tfrac{1}{2}y=\tan x\cos\theta,\quad \sin\tfrac{1}{2}y=\sin x\sin\phi,$$

$$\text{area CEDF}=\text{area ACED}+\text{area BCFD}-\text{area ACBD}$$
$$=2r^2\,(\pi-2\phi-2\theta\cos x);$$

hence the required average is

$$\int_0^{\frac{1}{2}\pi}\int_0^{2x}2r^2\,(\pi-2\phi-2\theta\cos x)\,r\,dx\,.\,2\pi r^2\sin y\,dy\div\int_0^{\frac{1}{2}\pi}\int_0^{2x}r\,dx\,.\,2\pi r^2\sin y\,dy$$

$$=\frac{4r^2}{\pi}\int_0^{\frac{1}{2}\pi}\int^{2x}(\pi-2\phi-2\theta\cos x)\,dx\sin y\,dy=16r^2\int_0^{\frac{1}{2}\pi}\sin^4\tfrac{1}{2}x\,dx=r^2\,(3\pi-8).$$

6059. (By W. J. C. Sharp, M.A.)—If tangents be drawn to each of the cubics $\lambda U+\mu H$, from its points of intersection with the line $\alpha x+\beta y+\gamma z=0$, prove that the points of contact all lie on the same quartic.

Solution by the Proposer.

Since the tangential of (x, y, z) is the intersection of

$$U_1\xi+U_2\eta+U_3\zeta=0 \text{ and } H_1\xi+H_2\eta+H_3\zeta=0$$

(Salmon's *Higher Plane Curves*, p. 152),

the quartic
$$\begin{vmatrix} U_1 & U_2 & U_3 \\ H_1 & H_2 & H_3 \\ \alpha & \beta & \gamma \end{vmatrix}=0$$

must pass through the points of contact of the tangents which can be drawn to $U=0$, from its points of intersection with $\alpha x+\beta y+\gamma z=0$, and from the form of the equation it is identical with the corresponding quartic for any cubic of the form $aU+bH=0$, including the Hessian itself.

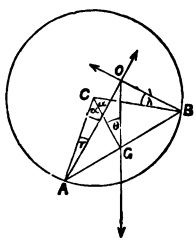
6776. (By W. H. H. Hudson, M.A.) — A uniform rod, of length $2a \sin \alpha$, is placed within a rough vertical circle, of radius a, and is on the point of motion, the coefficient of friction at its upper and lower ends being $\tan \lambda'$, $\tan \lambda$; prove that, if θ be the inclination to the vertical of the line joining the centre of the sphere to the centre of the rod,

$$\tan \theta = \frac{\sin (\lambda + \lambda')}{2 \cos (\alpha + \lambda) \cos (\alpha - \lambda')};$$

and examine the case where $\alpha + \lambda = \frac{1}{2}\pi$.

Solution by W. B. Grove, B.A.; D. Edwardes; *and others.*

Let C be the centre of the circle, AB the rod, O the point of intersection of the total resistances at A and B. From the triangle AOB,



$$\frac{\sin OBG}{\sin OAG} = \frac{\cos (\theta - OBG)}{\cos (\theta + OAG)},$$

or $$2 \cot OGB = \cot OAG - \cot BOG,$$

or $$2 \tan \theta = \tan (\alpha + \lambda) + \tan (\alpha - \lambda'),$$

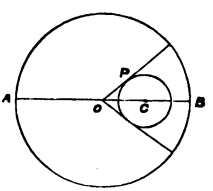
or $$2 \tan \theta = \frac{\sin (\lambda + \lambda')}{\cos (\alpha + \lambda) \cos (\alpha - \lambda')}.$$

If $\lambda = \frac{1}{2}\pi - \alpha$, the angle of friction at A is $\angle$ CAB. Hence the rod will rest in any position between the limits $\theta = 0$ and $\theta = \frac{1}{2}\pi$ inclusive.

6751. (By the Editor.) — A circle of given radius is drawn at random in a given circle; find the chance that a radius drawn at random in the fixed circle will cut the other.

Solution by Professor Matz, M.A.; Belle Easton; *and others.*

Let O be the centre of fixed, C the centre of variable, circle in any position; AOB the diameter through C; R and r the radii of these circles; OP a tangent; $\angle POB = \theta$; $OC = x$. Then, while C is fixed, the chance is $\frac{\theta}{\pi}$; hence, since $p = 1$, when $OC < r$, we have



$$p = \int_r^{R-r} \frac{\theta}{\pi} \cdot \frac{2\pi x\, dx}{\pi (R-r)^2} + \frac{r^2}{(R-r)^2}$$

$$= \frac{1}{\pi (R-r)^2} \left\{ \theta x^2 - \int x^2 d\theta \right\}_r^{R-r} + \frac{r^2}{(R-r)^2}$$

$$= \frac{1}{\pi} (\alpha + \cos \alpha \sin \alpha) + \tfrac{1}{2} \sin^2 \alpha, \text{ where } \sin \alpha = \frac{r}{R-r}.$$

If $r = \frac{1}{3}R$, $\alpha = \frac{1}{3}\pi$, and $p = 1$, as it should.

6507. (By Professor WOLSTENHOLME, M.A.)—If $a_1, a_2 \ldots, b_1, b_2 \ldots$ be all positive quantities, prove that

$$\int_0^\infty \frac{\sin a_1 x \sin a_2 x \ldots \sin a_n x \cos b_1 x \cos b_2 x \ldots \cos b_m x \sin \lambda x}{x^{n+1}}\, dx$$

will be $\frac{1}{2}\pi a_1 a_2 \ldots a_n$, if λ have any value not less than $\Sigma(a) + \Sigma(b)$.

[The value of the integral for all positive values of λ is positive, and increases as λ increases from 0 to the critical value above given. Thus the values of $\int_0^\infty \frac{\sin x \sin 2x \cos 3x \cos 4x \sin \lambda x}{x^3}\, dx$, when λ has the values 0, 1, 2 ... 10, are as 0, 7, 12, 15, 16, 17, 20, 24, 28, 31, 32; its value for all values of λ greater than 10 being π.]

Solution by the PROPOSER; C. B. S. CAVALLIN, M.A.; *and others.*

If $u = \sin a_1 x \sin a_2 x \ldots \sin a_n x \sin \lambda x$, it is obvious that the lowest power of x involved in u, when expressed as a series in ascending powers of x, is x^{n+1}; hence the quantities $\frac{4}{x^n}$, $\frac{\frac{du}{dx}}{x^{n-1}}$, $\frac{\frac{d^2u}{dx^2}}{x^{n-2}}$, &c. ... will all vanish when $x = 0$, and they will also vanish when $x = \infty$. Hence, by integrating by parts n times, we see that $\int \frac{u}{x^{n+1}}\, dx = \frac{1}{n!} \int_0^\infty \frac{\frac{d^n u}{dx^n}}{x}\, dx$.

Now $2 \sin \lambda x \sin a_1 x = \cos(\lambda - a_1) x - \cos(\lambda + a_1) x$,

$4 \sin \lambda x \sin a_1 x \sin a_2 x = \sin(\lambda - a_1 + a_2) x \sin(\lambda - a_1 - a_2) x$

$- \sin(\lambda + a_1 + a_2) x + \sin(\lambda + a_1 - a_2) x$,

$8 \sin \lambda x \sin a_1 x \sin a_2 x \sin a_3 x = \cos(\lambda - a_1 + a_2 - a_3) x$

$- \cos(\lambda - a_1 + a_2 + a_3) x + \ldots$,

and so on. In general, when n is even, $2^n \sin \lambda x \sin a_1 x \sin a_2 x \ldots \sin a_n x$ will be the sum of a number of terms each of which is of the form $\pm \sin(\lambda \pm a_1 \pm a_2 \ldots \pm a_n) x$, the sign of the term being regulated by the number of negative terms in the argument,—being positive when that number is odd and negative when that number is even if $\frac{1}{2}n$ be odd, but positive when that number is even and negative when that number is odd if $\frac{1}{2}n$ be even; when n is odd, $2^n u$ will be the sum of a number of terms of the form $\pm \cos(\lambda \pm a_1 \pm a_2 \ldots \pm a_n) x$, the sign of term being regulated as before with $\frac{1}{2}(n-1)$ in place of $\frac{1}{2}n$. Hence $\frac{d^n u}{dx^n}$ will in all cases be the sum of a number of terms of the form $\pm k^n \sin kx$, where k is one of the values of $(\lambda \pm a_1 \pm a_2 \ldots \pm a_n)$, and the sign of the term is + if the number of negative terms in the argument is even or zero, and − if this number is odd. Hence the integral is $\frac{1}{2}\pi \frac{1}{2^n n!} \times$ the sum of a number of terms each of which is of the form $\pm(\lambda \pm a_1 \pm a_2 \ldots \pm a_n)^n$, the rule for the pre-

fixed sign being the same as before when the coefficient $\lambda \pm a_1 \pm a_2 \ldots \pm a_n$ is positive, but the prefixed sign must be reversed when the coefficient is negative, since $\int_0^\infty \frac{\sin kx}{x}\,dx$ is $\pm \frac{1}{2}\pi$ according as k is positive or negative.

Hence, for all values of λ greater than $\Sigma(a)$, all these coefficients will be positive, and in that case the sum is $2^n n!\, a_1 a_2 \ldots a_n$, and the value of the integral is $\frac{1}{2}\pi\, a_1 a_2 \ldots a_n$. As λ decreases from $\Sigma(a)$ to 0, the form of the integral will change at each transit of λ through a value $-k$ which makes one of the coefficients vanish, each new form being deduced from the preceding by subtracting from the previous one the term $(-1)^p 2(\lambda + k)^n$, where p is the number of negative signs in k. I am not at present in a condition to prove that the integral always increases with λ from $\lambda = 0$ to $\lambda = \Sigma(a)$; but have found it to be so in all the cases tried.

Next, $2u \cos b_1 x = \sin a_1 x \ldots \sin a_n x [\sin(\lambda + b_1)x + \sin(\lambda - b_1)x] \equiv u_1 + u_2$; and, by the former part, $\int_0^\infty \frac{u_1}{x^{n+1}}\,dx = \int_0^\infty \frac{u_2 dx}{x^{n+1}} = \int_0^\infty \frac{u dx}{x^{n+1}}$, if $\lambda - b_1 > \Sigma(a)$, or if $\lambda > b_1 + \Sigma(a)$. Hence, under this condition,

$$\int_0^\infty \frac{\sin a_1 x \ldots \sin a_n x \cos b_1 x \sin \lambda x}{x^{n+1}}\,dx = \tfrac{1}{2}\pi\, a_1 a_2 \ldots a_n.$$

In the same manner it is obvious that the introduction of any number of factors $\cos b_2 x_1, \ldots \cos b_m x$ will not affect the value of the integral, provided λ has a value greater than $\Sigma(b) + \Sigma(a)$.

It has been tacitly assumed throughout that the symbols a, b always represent positive quantities.

[The theorem, assumed above for shortness' sake, that

$$\Sigma \pm (\lambda \pm a_1 \pm a_2 \ldots \pm a_n)^n$$

is equal to $2^n n!\, a_1 a_2 \ldots a_n$, is very readily proved. Since any one of the a symbols (a_1) occurs just as often with the positive as with the negative sign, and since the sign of the term depends upon the number of negative constituents, any two terms which differ only in the sign of a_1 will have opposite signs, and when $a_1 = 0$ will destroy each other. Hence the whole is divisible by each of the n factors $a_1, a_2 \ldots a_n$, and being of n dimensions can have no other factors. Also in each term the coefficient of $a_1 a_2 \ldots a_n$ is $+n!$ since the sign prefixed is $(-1)^p$ and each product $a_1 a_2 \ldots a_n$ is again multiplied by $(-1)^p$, p being the number of negative constituents. The number of terms is 2^n; hence the coefficient of $a_1 a_2 \ldots a_n$ is $2^n n!$; and the whole sum is $n!\, 2^n a_1 a_2 \ldots a_n$.

The sum of the 2^{m+n} terms, each of which is of the form

$$(-1)^p (x \pm a_1 \pm a_2 \ldots \pm a_n \pm b_1 \pm b_2 \ldots \pm b_m)^n,$$

where p is the number of negative a constituents, is $n!\, 2^{m+n} a_1 a_2 \ldots a_n$.]

5941. (By L. H. Rosenthal, M.A.)—Find, for the biquadratic whose roots are α, β, γ, δ, the equation whose roots are $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$, &c.

Solution by J. L. McKenzie, B.A.; Sarah Marks; *and others.*

Let the given biquadratic be $x^4+px^3+qx^2+rx+s=0$;

then $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=\Sigma(\alpha^3)-\delta^3-\frac{3s}{\delta}=-p^3+3pq-3r-\delta^3-\frac{3s}{\delta}$;

hence let $$y=-p^3+3pq-3r-x^3-\frac{3s}{x},$$

and eliminate x between this biquadratic and the given one. Or thus:—

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha),$$

and $$\alpha+\beta+\gamma=-p-\delta,\quad \alpha^2+\beta^2+\gamma^2=p^2-2q-\delta^2,$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q-\delta\left\{\Sigma(\alpha)-\delta\right\}=q+p\delta+\delta^2;$$

therefore put $$y=(-p-x)(p^2-2q-x^2-q-px-x^2),$$

or $$2x^3+3px^2+(p^2+3q)x+(3pq-p^3-y)=0;$$

and eliminate x between this and the given biquadratic. The work will be rather long; the resultant of a biquadratic and a cubic contains 76 terms.

5773 & 5950. (By J. L. McKenzie, B.A.)—Through any point P on a circular cubic, draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P.

Prove the following constructions to find the chord of curvature at a given point P on a circular cubic:—(1) Take any point A on the cubic, and draw a line through A parallel to the real asymptote, cutting the cubic in B. Let the tangent at P cut the cubic in C; draw AP cutting the cubic in D, CD in E, and BE in Q. Then Q is the point in which the circle of curvature at P cuts the cubic again. (2) Let the real asymptote cut the cubic in F; a line through P parallel to the asymptote, in G, and the tangent at P in C. Draw CG cutting the cubic in H, and HF in Q. Then, as before, PQ is the chord of curvature at P.

Solution by the Proposer.

The theorems in both of these questions may be proved by the method of "residuation."

5773. Consider the system of 7 points A, B, D, P, P, I, J (I and J being the circular points). The circle ABP and the line PD form one cubic through these points, and cutting the given cubic again in C and Q; the line QC must therefore pass through R, the "point-residual" of the 7 points. Again, the circle ABD and the line PP (the tangent at P), form another cubic through the 7 points, cutting the given cubic again in E and S; therefore ES passes through R. This proves the theorem.

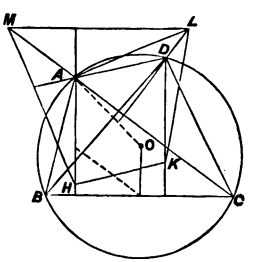
5950. (1) Let X be the point at infinity at which the asymptote touches the cubic. Take the system of 7 points on the cubic P, P, P, I, J, X, and

A. The circle of curvature at P and the line AX parallel to the asymptote form a cubic through these 7 points, cutting the given cubic again in B, and in the required point Q. Therefore BQ passes through E, the "point-residual" of the 7 points. Again, the tangent PP, the line at infinity XIJ, and the line CD form another cubic through the 7 points, cutting the given cubic again in C and D; therefore CD must pass through E. Hence the construction follows.

(2) Take now, as the 7 points, P, P, P, X, X, I, J; then the circle of curvature at P and the line XX (the asymptote) form a cubic through the 7 points, cutting the given cubic again in F and Q. And the tangent PP, the line PX parallel to the asymptote, and the line at infinity XIJ, form another cubic through the 7 points, cutting the given cubic again in C and G. Hence the lines FQ and CG must both meet the given cubic in the same point H,—the "point-residual" of the 7 points.

6403. (By D. Edwardes.)—If ABCD be a quadrilateral inscribed in a circle, and H, K, L, M are the orthocentres of the triangles formed by its sides and diagonals; prove that, (1) HKLM is a quadrilateral equal and similar to ABCD; (2) A, B, C, D are the orthocentres of the triangles formed from the sides and diagonals of HKLM; (3) the lines joining similar angular points meet in a point and mutually bisect one another; and (4) that this point bisects also the line joining the centres of the circles ABCD and HKLM.

Solution by C. Morgan, B.A.; G. F. Walker, M.A.; *and others.*



1. The distances AH and DK are each double of the distance of O from BC (O being the centre of the circle ABCD); hence, AH and DK being equal and parallel, HD, HK are equal and parallel. Similarly for the other sides of the quadrilateral. Hence the quadrilateral HKLM is equal in all respects to ABCD.

2. Since HM is parallel to CD, AL, which is perpendicular to DC, is also perpendicular to MH. Similarly, AH is perpendicular to ML, therefore A is orthocentre of HLM; and so for the other triangles formed by the sides and diagonals of HKLM.

3. Since AHKD is a parallelogram, AK and HD bisect each other. Similarly, since CHMD is a parallelogram, MC and HD bisect each other, therefore AK and MC bisect each other at the middle point of HD; and similarly any pair of lines joining corresponding vertices bisect each other, and must therefore all meet in one point, and bisect each other.

4. This point is evidently, then, a centre of similitude (internal) of the two equal circles ABCD, HKLM, and therefore bisects the distance between their centres.

6627. (By Professor Townsend, F.R.S.)—Two shallow circular arcs, of arbitrary versed-sines, being supposed described on either segment, and on the same produced outwards to half its length, of a uniform elastic rod, supported in a horizontal position by one central and two terminal props, and bent slightly by its own weight; show that the vertical depression varies as the product of the ordinates to the arcs throughout the entire extent of the segment.

Solution by the Proposer.

Taking the horizontal and vertical lines through the centre of the rod, in the vertical plane of its equilibrium, as axes of x and of y respectively, and, in the integral of the familiar differential equation of the curve of strain of its axis under the action of gravity, viz., $\frac{d^4y}{dx^4} = k$, (where k is a constant depending on its density, elasticity, and transverse section,) determining the values of the four arbitary constants from the conditions that $y = 0$ and $\frac{dy}{dx} = 0$ for $x = 0$, and that $y = 0$ and $\frac{d^2y}{dx^2} = 0$ for $x = a$, where a is its semi-length; we get without difficulty, for the equation in finite terms of its strained axis, $y = kx^2(a-x)(\frac{5}{2}a-x)$; from which the property immediately follows, the ordinates to the two arcs, whatever be their versed-sines, varying as the two products $x(a-x)$ and $x(\frac{5}{2}a-x)$ in the two cases respectively; and therefore, &c.

5939. (By R. A. Roberts, M.A.) — In the motion of a material particle constrained to move on a sphere, and attracted according to the law of the inverse fifth power of the distance by a uniform circular plate whose circumference is on the sphere, if the velocity in any position be that from infinity under the action of the force, prove that the orbit will be a circle orthogonal to the circumference of the plate.

Solution by the Proposer.

For the potential V of the plate at any point of space, we have $V = \frac{1}{4}\frac{m}{(\mu^2-c^2)(\mu^2-\nu^2)}$, where μ is half the sum and ν half the difference of the greatest and least distances from the circumference of the plate. Hence, at any point of a sphere passing through the circumference of the plate, V varies inversely as the square of the distance from the plane of the plate. But, if v be the velocity of the particle, v^2 varies as V, and $\int v ds$ is a minimum; or, using polar coordinates on the sphere, we have $\int \frac{(d\theta^2 \cdot \sin^2\theta d\phi^2)^{\frac{1}{2}}}{\cos\alpha - \cos\theta}$ a maximum, giving $\frac{\sin^2\theta}{\left\{\frac{d\theta^2}{d\phi^2} + \sin^2\theta\right\}^{\frac{1}{2}}} = C(\cos\alpha - \cos\theta)$, which represents a circle cutting the circle $\theta = \alpha$ orthogonally.

This theorem is also true, if we substitute for the plate the spherical surface bounded by a small circle.

[Mr. ROBERTS remarks that Mr. SHARP's solution of Quest. 5984 (*Reprint*, XXXII., p. 42) is altogether incorrect; since a value of V has been therein obtained different from what is given above, and that the process, furthermore, makes the resultant pass through the pole of the plane of the plate with regard to the sphere, which is not the case.]

6185. (By E. W. SYMONS, B.A.)—The vertex of a parabola moves along the pedal of a given curve, while its focus is fixed at the pole; prove that its envelop is the first negative pedal of the given curve.

Solution by H. HAYASH, M.A.; E. RUTTER; *and others.*

We begin by stating the known theorem, that the envelop of circles described on the radii vectores of a given curve is the pedal of that curve, thus,—the envelop of a circle passing through the pole, and whose centre lies on the radius vector of a given curve at a distance from the pole equal to half the length of that radius vector, is the pedal of the given curve, *i.e.*, the locus of foot of perpendicular let fall from the pole on any tangent to the given curve.

Reciprocation of this with respect to the pole gives the following theorem: The envelop of a parabola with focus at the pole, directrix parallel to any tangent to a given curve, at double the distance from the pole that the said tangent is, will be the envelop of a line drawn through the point of contact of the tangent and perpendicular to the line joining that point of contact to the pole, *i.e.*, will be the first negative pedal. Stated in different words, this is the theorem enunciated.

6640. (By R. RAWSON.)—Assuming that

$$u = \{\phi_1(x)\,\theta(x, y) + 1\}^2 - \phi_2(x) = (x - y_1)(x - y_2) \ldots (x - y_n) \ldots\ldots (1),$$

where u, $\phi_1(x)$, $\theta(x, y)$, and $\phi_2(x)$ are rational in x; show that

$$\int \frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\,[\phi_2(y_1)]^{\frac{1}{2}}} + \ldots\ldots \int \frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\,[\phi_2(y_n)]^{\frac{1}{2}}} = \frac{1}{x} \Big| \frac{\epsilon\psi(x)}{\phi_1(x)\,[\phi_2(x)]^{\frac{1}{2}}} \times \log \frac{\phi_1(x)\,\theta(x, y) + 1 - [\phi_2(x)]^{\frac{1}{2}}}{\phi_1(x)\,\theta(x, y) + 1 + [\phi_2(x)]^{\frac{1}{2}}} \ldots\ldots\ldots (2),$$

where $\epsilon = \mp 1$, $x \mid f(x)$ is used as a convenient symbol to represent the coefficient of x in the development of $f(x)$, and $x^{-1} \mid f(x)$ is the coefficient of x^{-1} in the development of $f(x)$, &c. [The $\psi(x)$ may be either rational or irrational; when $\psi(x)$ is rational, and $\phi_1 x = x - a$, the above integral becomes that determined by ABEL's theorem.]

Solution by the PROPOSER.

Differentiating (1), we have

$$dx = -2\left\{\phi_1(x)\,\theta(x,y)+1\right\}\phi_1(x)\frac{d}{dy}\theta(x,y)\,dy \div \frac{du}{dx}$$

$$= \pm 2\,\phi_1(x)\left\{\phi_2(x)\right\}^{\frac{1}{2}}\frac{d}{dx_1}\theta(x,y) \div \frac{du}{dx},$$

$$\therefore \quad \frac{\psi(x)\,dx}{\phi_1(x)\left\{\phi_2(x)\right\}^{\frac{1}{2}}} = \frac{2\epsilon\,\psi(x)\frac{d}{dy}\theta(x,y)}{\frac{d}{dx}\mathrm{F}(x,y)}, \text{ when } u = \mathrm{F}(x,y)\ldots\ldots\ldots(3).$$

Now (3) is true for every value of x which satisfies (1). Hence, by integrating (3), and putting for x the values $y_1, y_2, \ldots y_n$ successively, there results

$$\int\frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\left\{\phi_2(y_1)\right\}^{\frac{1}{2}}} + \ldots \int\frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\left\{\phi_2(y_n)\right\}^{\frac{1}{2}}}$$

$$= 2\epsilon\int\left\{\frac{\psi(y_1)\frac{d}{dy}\theta(y_1,y)}{\frac{d}{dy_1}\mathrm{F}(y_1,y)} + \ldots \frac{\psi(y_n)\frac{d}{dy}\theta(y_n,y)}{\frac{d}{dy_n}\mathrm{F}(y_n,y)}\right\}dy + \mathrm{C}\ldots\ldots(4).$$

The right-hand side of (4) is a particular case of the property enunciated in Quest. 6559, where $m = 0$, $f(x) = 1$, and $\psi(x, y) = \psi(x)$. Therefore equation (4) becomes

$$\int\frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\left\{\phi_2(y_1)\right\}^{\frac{1}{2}}} + \ldots \int\frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\left\{\phi_2(y_n)\right\}^{\frac{1}{2}}}$$

$$= \frac{1}{x}\Big|\ 2\epsilon\int\frac{\psi(x)\frac{d}{dy}\theta(x,y)}{\mathrm{F}(x,y)}\,dy + \mathrm{C} = \frac{1}{x}\Big|\ 2\epsilon\,\psi(x)\int\frac{d\theta(x,y)}{\mathrm{F}(x,y)} + \mathrm{C}$$

$$= \frac{1}{x}\Big|\ \epsilon\,\frac{\psi(x)}{\phi_1(x)\left\{\phi_2(x)\right\}^{\frac{1}{2}}}\ \log\frac{\phi_1(x)\,\theta(x,y)+1-\left\{\phi_2(x)\right\}^{\frac{1}{2}}}{\phi_1(x)\,\theta(x,y)+1+\left\{\phi_2(x)\right\}^{\frac{1}{2}}} + \mathrm{C},$$

which proves the relation in the Question. The above demonstration assumes that the roots of (1) are unequal. ABEL has attempted to prove his theorems when some of the roots are equal, but his results appear to me to require reconsideration.

6500. (By G. F. WALKER, M.A.) — A chord PQ of the parabola $y^2 = 4ax$ passes through a fixed point $(c, 0)$ on the axes. Circles are described touching the parabola at P and Q, and passing through the focus; show that their second common point must lie on the curve

$$(x^2+y^2-ax)^2 - 3c\,(x-a)\,(x^2+y^2-ax) - \frac{c^2}{a}(3a-c)\,y^2 = 0.$$

Solution by I. H. TURRELL, M.A.; *the* PROPOSER; *and others.*

The equation to a circle touching at the point (x, y), and passing through the focus, is easily found to be

$$(1+m_1^2)(y^2-4ax) = \left(y-m_1x-\frac{a}{m_1}\right)(y+m_1x+3am_1),$$

or
$$m_1^2(x^2+y^2)-ax(3+m_1^2)-ay\left(3m_1-\frac{1}{m_1}\right)+3a^2=0.$$

The circle touching at m_2, in same manner, is

$$m_2^2(x^2+y^2)-ax(3+m_2^2)-ay\left(3m_2-\frac{1}{m_2}\right)+3a^2=0;$$

and we want to eliminate m_1, m_2 from these and the condition $m_1m_2=-\frac{a}{c}$.

Subtracting and dividing by m_1-m_2, we get

$$(m_1+m_2)(x^2+y^2-ax)-ay\left(3+\frac{1}{m_1m_2}\right)=0 \quad \ldots\ldots\ldots\ldots (1).$$

Multiply first by m_2^2, and subtract from the second multiplicand by m_1^2, and we get, after division by m_1-m_2,

$$-3ax(m_1+m_2)-3aym_1m_2+\frac{ay(m_1^2+m_1m_2+m_2^2)}{m_1m_2}+3a^2(m_1+m_2)=0,$$

or
$$\frac{ay(m_1+m_2)^2}{m_1m_2}-3a(x-a)(m_1+m_2)-ay(3m_1m_2+1)=0.$$

Substituting from (1) for the last term, and dividing out by m_1+m_2, we get
$$\frac{ay(m_1+m_2)}{m_1m_2}-3a(x-a)-m_1m_2(x^2+y^2-ax)=0 \quad \ldots\ldots\ldots\ldots (2).$$

Eliminating m_1+m_2 between (1) and (2), we get

$$\frac{a^2y^2}{m_1m_2}\left(3+\frac{1}{m_1m_2}\right)=(x^2+y^2-ax)[m_1m_2(x^2+y^2-ax)+3a(x-a)];$$

or, remembering $m_1m_2=-\frac{a}{c}$, we find

$$-acy^2\left(3-\frac{c}{a}\right)=(x^2+y^2-ax)\left[-\frac{a}{c}(x^2+y^2-ax)+3a(x-a)\right],$$

$$c^2y^2(3a-c)=a(x^2+y^2-ax)[x^2+y^2-ax-3c(x-a)].$$

[The locus of the centre of either circle is $27ay^2=(2x-a)(5a-x)^2$].

6735. (By Prince CAMILLE DE POLIGNAC.)—An unclosed polygon is inscribed in a conic and circumscribed about another; M_1, M_2 are two consecutive fixed sides; a_1, a_2 any other pair of consecutive sides taken in the same order; a_1 meets M_2 in m_2; and a_2 meets M_1 in m_1. If the line m_1m_2 passes through a fixed point, prove that the conics have double contact.

Solution by the PROPOSER.

Let any conic, and two tangents thereto, be

$$xy-z^2=0\equiv S,\quad x+\mu^2y-2\mu z=0,\quad x+\mu'^2y-2\mu' z=0\ldots\ldots(1,2,3);$$

then, tangent μ meets the axis of y in the point $(y=0,\ x=2\mu z)$; and tangent μ' meets the axis of x in the point $(x=0,\ y=2\mu'^{-1}z)$; and the condition that the line joining these two points should pass through a fixed point (α,β,γ) is $\begin{vmatrix}\alpha & \beta & \gamma\\ 2\mu & 0 & 1\\ 0 & 2 & \mu'\end{vmatrix}=0$, or $\beta\mu\mu'-2\gamma\mu+\alpha=0$ .....(4).

Eliminating μ, μ' between (2), (3), (4), we get, for the locus of the intersection of tangents,

$$4\gamma^2xy+(\beta x+\alpha y)(\beta x+\alpha y-4\gamma z)=0,\text{ or } S'\equiv 4\gamma^2S+(\beta x+\alpha y-2\gamma z)^2=0,$$

a conic which passes through the point (x,y), and has double contact with S.

Now, in virtue of equation (4), tangents μ, μ' may be considered as two consecutive sides of the polygon; the same is true of x and y, which correspond respectively to the two consecutive values $\mu=0$, $\mu'=\infty$.

The polar of (α,β,γ) is the chord of contact $\beta x+\alpha y-2\gamma z=0$; hence the tangents common to S and S′ meet in (α,β,γ).

The lines through (α,β,γ) answering to each pair of consecutive sides trace out on the axes a homographic system of points, and the tangents through (α,β,γ) meet the axes in the foci of the system. For these are given by the equation $\beta\mu^2-2\gamma\mu+\alpha=0$, which expresses the condition that the point (α,β,γ) should lie on the tangent $x+\mu^2y-2\mu z=0$.

The anharmonic ratio of four consecutive tangents, or of four consecutive legs of the anharmonic pencil through (α,β,γ), is a constant quantity $=\dfrac{\alpha\beta}{4\gamma^2}$.

When the foci coincide, $\alpha\beta=\gamma^2$; the point (α,β,γ) lies on the conic S, and the two conics have a contact of the third order.

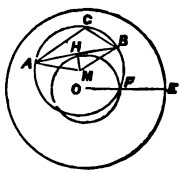
When the double contact is real, it becomes obvious, on drawing the figure, that the process of inscribing tangents converges towards the common tangents. In other words, the homographic points on the axes converge towards the foci.

This may be otherwise stated. The successive sides of the polygon correspond to the powers of a linear substitution, involved in the homographic equation (4). Hence the nth power of any linear substitution (to real foci) converges ultimately towards one focus, and the nth power of the inverse substitution towards the other.

1843. (By the EDITOR.)—(1) Three points being taken at random within a circle, prove that the chance that the circle circumscribing the triangle formed by joining them lies wholly within the given circle, is $\frac{3}{4}$; and (2) four points being taken at random within a sphere, prove that the chance that the sphere circumscribing the tetrahedron formed by joining them lies wholly within the given sphere, is $\dfrac{48\pi^2}{1925}$.

Solution by Professor SEITZ, M.A.

1. Let O be the centre of the given circle, ABC the triangle formed by joining the three random points A, B, C, and M the centre of the circle circumscribing ABC. From O as centre, with a radius OF equal to the difference between OE and AM, draw a circle.



Now, while AB and AC are given in length and direction, AM being less than OE, the area of the circle OF represents the number of ways the three points can be taken, so that the circle circumscribing the triangle will lie wholly within the given circle.

Let AB = $2x$, OE = r, $\angle$AMH = θ, $\angle$BAC = ϕ, and ψ = the angle which AB makes with a fixed line. Then we have

$$AM = x \operatorname{cosec} \theta, \quad OF = r - x \operatorname{cosec} \theta, \quad AC = 2x \sin(\theta - \phi) \operatorname{cosec} \theta;$$

an element of the circle at B is $4x\,dx\,d\psi$, and at C it is

$$d.(AC) \sin(\theta-\phi) \,.\, 2x \operatorname{cosec} \theta \, d\phi = 4x^2 \operatorname{cosec}^3 \theta \sin\phi \sin(\theta-\phi)\, d\theta\, d\phi.$$

The limits of θ are 0 and π, and doubled, since C may lie on either side of AB; those of ϕ are 0 and θ; of x, 0 and $r \sin\theta = x'$; and of ψ, 0 and 2π.

Hence, since the whole number of ways the three points can be taken is $\pi^3 r^6$, the chance in question is

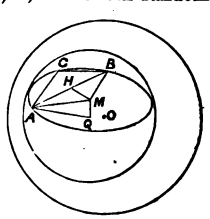
$$\frac{2}{\pi^3 r^6} \int_0^{\pi} \int_0^{\theta} \int_0^{x'} \int_0^{2\pi} \pi (r - x \operatorname{cosec}\theta)^2 \operatorname{cosec}^3\theta \, d\theta \sin\phi \sin(\theta-\phi)\, d\phi \,.\, 16x^3\, dx\, d\psi$$

$$= \frac{64}{\pi r^6} \int_0^{\pi} \int_0^{\theta} \int_0^{x'} (r - x \operatorname{cosec}\theta)^2 \operatorname{cosec}^3\theta \, d\theta \sin\phi \sin(\theta-\phi)\, d\phi\, x^3\, dx$$

$$= \frac{16}{15\pi} \int_0^{\pi} \int_0^{\theta} \sin\theta \, d\theta \sin\phi \sin(\theta-\phi)\, d\phi$$

$$= \frac{8}{15\pi} \int_0^{\pi} (\sin\theta - \theta\cos\theta) \sin\theta \, d\theta = \frac{2}{3}.$$

2. Let O be the centre of the given sphere, A, B, C, D the four random points, Q the centre of the sphere whose surface contains A, B, C, D, and M the centre of the small circle of this sphere, which passes through A, B, C. Draw MH perpendicular to AB.



Now, while AB, AC, AD are given in length and direction, AQ being less than the radius of the given sphere, the volume of a sphere whose radius is equal to the difference of the given radius and AQ, represents the number of ways the four points can be taken, so that the sphere QA will lie wholly within the given sphere.

Let AB = $2x$, $\angle$AMH = θ, $\angle$BAC = ϕ, $\angle$AQM = ψ, μ = the angle AB makes with a fixed line, and ω = the angle ABC makes with a plane through AB. Then we have

$$AM = x \operatorname{cosec}\theta, \quad AQ = x \operatorname{cosec}\theta \operatorname{cosec}\psi, \quad AC = 2x \sin(\theta-\phi) \operatorname{cosec}\theta;$$

an element of the sphere at B is $16\pi x^2 \sin\mu\, dx\, d\mu$, at C it is

$$d\,.\,(\mathrm{AC})\sin(\theta-\phi)\,.\,2x\,\mathrm{cosec}\,\theta\,d\phi\,.\,\mathrm{AC}\sin\phi\,d\omega$$
$$= 8x^3\,\mathrm{cosec}^4\theta\sin^2\phi\sin^2(\theta-\phi)\,d\theta\,d\phi\,d\omega,$$

and at D it is d (spher. seg. ABCD) $= \pi x^3\,\mathrm{cosec}^3\theta\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2 d\psi$.

The limits of θ are 0 and π, and doubled, since C may lie on either side of AB; those of ϕ are 0 and θ; of ψ, 0 and π, and doubled, since D may lie on either side of the plane ABC; those of x are 0 and $r\sin\theta\sin\psi = x'$; of μ, 0 and π; and of ω, 0 and 2π. Hence, since the whole number of ways the four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the chance in question is

$$\frac{4}{(\frac{4}{3}\pi r^3)^4}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'}\int_0^{\pi}\int_0^{2\pi}\tfrac{4}{3}\pi\,(r-x\,\mathrm{cosec}\,\theta\,\mathrm{cosec}\,\psi)^3\,\mathrm{cosec}^7\theta\,d\theta\sin^2\phi$$
$$\times\sin^2(\theta-\phi)\,d\phi\,.\,8\pi\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2 d\psi\,.\,16\pi x^2 dx\sin\mu\,d\mu\,d\omega$$
$$= \frac{864}{r^{12}}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'}(r-x\,\mathrm{cosec}\,\theta\,\mathrm{cosec}\,\psi)^3\,\mathrm{cosec}^7\theta\,d\theta\sin^5\phi\sin^2(\theta-\phi)\,d\phi$$
$$\times\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2 d\psi\,x^3 dx$$
$$= \frac{24}{55}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\sin^2\theta\,d\theta\sin^2\phi\sin^2(\theta-\phi)\,d\phi\,(1-\cos\psi)^2\sin^5\psi\,d\psi$$
$$= \frac{1024}{1925}\int_0^{\pi}\int_0^{\theta}\sin^2\theta\,d\theta\sin^2\phi\sin^2(\theta-\phi)\,d\phi$$
$$= \frac{128}{1925}\int_0^{\pi}(39-2\theta\sin^2\theta-3\sin\theta\cos\theta)\sin^2\theta\,d\theta = \frac{48\pi^2}{1925}.$$

[Other Solutions are given in *Reprints*, Vol. VIII., pp. 90—92; XIII., 17—21, 95—99; XVI., 50—53.]

6830. (By Professor Crofton, F.R.S.)—Prove that

$$e^{\mathrm{D}+x^{-1}}\,\mathrm{F}\,(x) = \frac{x+1}{x}\,\mathrm{F}\,(x+1).$$

Solution by J. J. Walker, M.A.

The formula of the Question may be generalized; viz., a, b being any constants, $\exp.\, a\left\{\mathrm{D}+(x+b)^{-1}\right\}\mathrm{F}\,(x) = \left\{1+a\,(x+b)^{-1}\right\}\mathrm{F}\,(x+a)$.
To verify this, let v be a function of x satisfying $v^2+a\mathrm{D}v = 0$; then, r being any integer, $(a\mathrm{D}+v)^r = a^r\mathrm{D}^r + ra^{r-1}v\mathrm{D}^{r-1}$. For, assuming this to hold for any value of r, it is easily shown that it holds for $r+1$. Hence, immediately, $f(a\mathrm{D}+r) = f(a\mathrm{D})+vf'(a\mathrm{D})$; and in particular

$$\exp.\,(a\mathrm{D}+v)\,\mathrm{F}\,(x) = (1+v)\exp.\,(a\mathrm{D})\,\mathrm{F}\,(x) = (1+v)\,\mathrm{F}\,(x+a).$$

But $v^2+a\mathrm{D}v = 0$ gives $\quad v = a\,(x+b)^{-1}$.

[Mr. THOMAS and others prove the theorem thus :—

$(D+x^{-1})\,x^n = x^{-1}\,Dx^{n+1}$, and $(D+x^{-1})^m\,x^n = x^{-1}\,D^m\,x^{n+1}$;

therefore $f(D+x^{-1})\,\phi(x) = x^{-1} f(D)\,x\,\phi(x)$,

therefore $e^{D+x^{-1}}\,F(x) = x\quad e^D\,x\,F(x) = \frac{x+1}{x}\,F(x+1)$.]

6795 & 6826. (By Professor SYLVESTER, F.R.S.)—(6795) Prove that

$1 + 1\,.\,m + 1\,.\,5\,\frac{m(m-1)}{1} + 1\,.\,5\,.\,9\,\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} + \ldots$ is divisible by 2^m. Ex.: $1+5+10\,(5)+10\,(5\,.\,9)+5\,(5\,.\,9\,.\,13)+5\,.\,9\,.\,13\,.\,17$

$= 1+5+50+450+2925+9945 = 13376 = 2^5 \times 418.$

(6826.) Find the greatest common measure of D_n and D_{n+1}, where D_n represents a determinant of the n^{th} order of the form

$$\begin{vmatrix} 1 & 1 & . & . & . & . \\ 1 & 3 & 2 & . & . & . \\ . & 1 & 5 & 3 & . & . \\ . & . & 1 & 7 & 4 & . \\ . & . & . & 1 & 9 & 5 \\ . & . & . & . & 1 & 11 \end{vmatrix}$$

Solution by W. J. C. SHARP, M.A.; E. BUCK, B.A.; *and others.*

The determinants in 6826 satisfy the equation of differences

$$D_{n+1} = (2n+1\;\; D_n - n\,D_{n-1};$$

so that every measure of D_{n+1} and D_n measures D_{n-1}, and therefore the G. C. M. of D_{n+1} and D_n can differ frcm that of D_n and D_{n-1} by involving an additional factor which measures n. But every measure of n and D_{n+1} must measure the determinant obtained by writing

$$\begin{matrix} . & . & . & . & . & 1 & . & -1 & . & 0 \\ . & . & . & . & . & 0 & . & 1 & . & 1 \end{matrix}$$

instead of the last two lines of D_{n+1} and this $= -D_{n-1}$; therefore every measure of D_{n+1} and n measures D_{n-1}, and the G. C. M. of D_{n+1} and D_n is that of D_n and D_{n-1}, and ultimately of D_3 and D_2 or 2.

Again, denoting the series given in 6795 by u_m, we have

$$u_{m+1} - 2u_m = -1\,.\,m - 1\,.\,5\,\frac{m(m-3)}{1\,.\,2} - 1\,.\,5\,.\,9\,\frac{m(m-1)(m-5)}{3!} - \&c.$$

$$= -m\left[1 + 1\,.\,5\left\{\frac{m-1}{2} - 1\right\} + 1\,.\,5\,.\,9\left\{\frac{(m-1)(m-2)}{3!} - \frac{m-1}{2!}\right\}\right.$$

$$\left. + 1\,.\,5\,.\,9\,.\,13\left\{\frac{(m-1)(m-2)(m-3)}{4!} - \frac{(m-1)(m-2)}{3!}\right\} + \&c.\right]$$

Denoting by P the common value of the two terminal pressures, and by Q that of the central pressure, by p the common diminution in length of the two terminal props, and by q that of the central prop; then since, on elementary principles of longitudinal elasticity,

$$p : h = P : E, \quad q : h = Q : E,$$

and since evidently $2P + Q = W$, therefore

$$E(q-p) = (Q-P)h = \tfrac{1}{2}(3Q-W)h;$$

from which it appears that Q, and therefore P, would be known if $(q-p)$ were connected with it by another linear relation of the same form, which it may be shown it is as follows.

Since, on known principles of transverse deflection, $(q-p)$ cannot exceed positively $\frac{5}{48} \cdot \frac{l^3}{K} \cdot W$, which would be its value if the central prop were removed, in which case Q would = 0, and cannot exceed negatively $\frac{3}{48} \cdot \frac{l^3}{K} \cdot W$, which would be its value if the two terminal props were removed, in which case Q would = W; therefore, for every intermediate value of $(q-p)$, the corresponding value of Q, in accordance with the general principle that, under the same circumstances of production, all small effects are proportional to their causes, will be given by the proportion

$$Q : W = \left[\frac{5}{48} \cdot \frac{l^3}{K} \cdot W - (q-p)\right] : \frac{8}{48} \cdot \frac{l^3}{K} \cdot W;$$

which accordingly is the second linear relation connecting the two quantities in question.

Eliminating $(q-p)$ between the two linear equations connecting it with Q, we get at once for Q the value

$$\left[\frac{1}{3} \cdot \frac{l^3}{K} + 3\frac{h}{E}\right] Q = \left[\frac{10}{48} \cdot \frac{l^3}{K} + \frac{h}{E}\right] W;$$

and, since $P = \frac{1}{2}(W-Q)$, consequently for P the value

$$\left[\frac{1}{3} \cdot \frac{l^3}{K} + 3\frac{h}{E}\right] P = \left[\frac{3}{48} \cdot \frac{l^3}{K} + \frac{h}{E}\right] W;$$

which are manifestly identical with those of the question.

In the extreme case when $K = \infty$, that is, when the supported rod is perfectly rigid, then $P = Q = \frac{1}{3}W$; which are the known values of P and Q in that case.

In the other extreme case when $E = \infty$, that is, when the supporting props are perfectly rigid, then $P = \frac{3}{16}W$, and $Q = \frac{10}{16}W$; which are their known values in that case also.

6763. (By the Rev. C. Taylor, D.D.)—If the axes of one rectangular hyperbola are parallel to the asymptotes of another, and the centre of each lies on the other; prove that an infinity of circles can be drawn through the centre of either so as to meet the other again in conjugate triads with respect to the former.

Solution by G. F. WALKER, M.A.; E. HAIGH, B.A.; *and others.*

The equations to the two hyperbolas may be written

$$x^2-y^2 = a^2, \quad 2[xy-ay\sec\theta-ax\tan\theta] = 0,$$

and, taking λ so that $\lambda(x^2-y^2-a^2)+2[xy-ay\sec\theta-ax\tan\theta] = 0$ may represent two straight lines, we have

$$\lambda^3a^2+2a^2\tan\phi\sec\phi-\lambda a^2\sec^2\phi+\lambda a^2\tan^2\phi+\lambda a^2 = 0,$$

or

$$\lambda^3a^2+2a^2\tan\phi\sec\phi = 0.$$

Hence, for these two conics, the invariants Θ and Θ' both vanish, and conjugate triads with respect to either can be found on the other. But the circle round a conjugate triad with respect to a rectangular hyperbola passes through the centre, therefore, &c.

6829. (By Professor WOLSTENHOLME, M.A.)—Given a fixed point S and a fixed straight line, SX is drawn perpendicular to the line and bisected in O; p being any point in the plane, pk is let fall perpendicular on the line, and kO, pS meet in P. Prove that (1) the relation between p, P is reciprocal; (2) whatever curve be traced out by p, the locus of P will be a curve of the same order and class; (3) the tangents at p, P to their loci intersect always on the fixed straight line; (4) the loci of p, P will coincide when it is a conic of which S and the given straight line are focus and corresponding directrix.

Solution by Prof. GENESE, M.A.; T. WOODCOCK, B.A.; *and others.*

1. Let Sp meet Xk at R. Since SO = OX, the pencil $k(\dot{\mathrm{X}}\mathrm{O}\dot{\mathrm{S}}p)$ is harmonic, hence the range $(\dot{\mathrm{R}}\mathrm{P}\dot{\mathrm{S}}p)$ is so. Thus, the relation between p and P is reciprocal.

2. Let $\mathrm{P_1P_2}$, p_1p_2 meet in Q. Then $\mathrm{Q}(\dot{\mathrm{R}}_1\mathrm{P}_1\dot{\mathrm{S}}p_1)$ is harmonic. Therefore to every point of the locus of p on Qp, will correspond a point on $\mathrm{QP_2}$, and *vice versa;* the degree of the loci of p and P must be the same. Also, since to any line through p_1 there corresponds only one straight line through $\mathrm{P_1}$, and *vice versa*, the number of tangents from p_1 to the locus of p must equal the number of tangents from $\mathrm{P_1}$ to the locus of P, *i.e.*, the loci are of the same class.

3. Let p_1, p_2 be two positions of p; $\mathrm{P_1}$, $\mathrm{P_2}$, $\mathrm{R_1}$, $\mathrm{R_2}$ corresponding positions of P and R. Since $(\dot{\mathrm{R}}_1\mathrm{P}_1\dot{\mathrm{S}}p_1) = (\dot{\mathrm{R}}_2\mathrm{P}_2\dot{\mathrm{S}}p_2)$, therefore $\mathrm{R_1R_2}$, $\mathrm{P_1P_2}$, p_1p_2 cointersect; that is, the join of two points meets the join of the corresponding points on the given straight line. If p_1, p_2 be consecutive points on the locus of p, we have the theorem (3); and (4) is evident from (1). The correspondence can be exhibited very clearly by means of biangular coordinates. Since SXR is a right angle, XS bisects angle pSP. Thus, X, S being the poles, if (α = cot SXp, β = cot pSX) be the coordinates of p, x, and (x, y) the coordinates of P, then $x = -\alpha$, $y = \beta$.

[More generally, the loci will coincide when it is a conic in which S and the given straight line are pole and polar.]

6804. (By the Rev. W. A. Whitworth, M.A.)—The sides of a triangle are expressed by integers in arithmetical progression; if each side be increased by 50, the radius of the inscribed circle will be increased by 17; but, if each side be increased by 60, the radius of the inscribed circle will be increased by 20; find the triangle.

Solution by Kate Gale; E Buck, B.A.; *and others.*

Let the sides be $a-b$, a, $a+b$; then $r = (2\sqrt{3})^{-1}(a^2-4b^2)^{\frac{1}{2}}$,

therefore $r+17 = (2\sqrt{3})^{-1}\{(a+50)^2-4b^2\}^{\frac{1}{2}}$,

similarly $r+20 = (2\sqrt{3})^{-1}\{(a+60)^2-4b^2\}^{\frac{1}{2}}$;

and from these equations we get $a = 26$, $b = 11$, $r = 4$.

6112. (By F. Morley, B.A.)—The lengths of parallel straight lines intercepted between the angles of a tetrahedron and the opposite faces are p, q, r, s, reckoned of different signs as they fall on a face or a face produced; prove that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s} = 0$, and deduce the corresponding theorem for a triangle.

Solution by G. Turriff, M.A.; E. W. Symons, B.A.; *and others.*

Let OABC be the tetrahedron, and let OD, AE, BF, CG be drawn parallel to one another through the vertices, meeting the opposite faces or these produced in D, E, F, G. If OD meet the face ABC, the other lines meet the faces produced. Take OA, OB, OC for oblique axes, and let the equations of ABC and OD be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1, \quad \frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = r.$$

If OD $= p$, we get $\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c} = \frac{1}{p}$,

and evidently $\frac{1}{AE} = \frac{1}{q} = -\frac{\lambda}{a}$, $\frac{1}{BF} = \frac{1}{r} = -\frac{\mu}{b}$ and $\frac{1}{CG} = \frac{1}{s} = -\frac{\nu}{c}$,

therefore $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s} = 0$.

If the lines are drawn parallel to a face, one line becomes infinite and its reciprocal vanishes, and we have the case of the triangle.

6816. (By Dr. Macfarlane, F.R.S.E.)—A colony is formed by N persons of different races, and the two sexes are equally numerous. The colony remains unaffected by either immigration or emigration. What is the least time in which it can become homogeneous?

Solution by the PROPOSER.

Let p_1 denote the probability of a person dying childless; p_2 the probability of a person having children but no grandchildren; p_3 of having grandchildren but no great-grandchildren; and so on. Let k denote the average number of children for a person who has children. In order that the colony may become homogeneous in the least time, there must be no intermarriages between kindred. Under that condition, $p_2 = p_1^k$, $p_3 = p_1^{k^2}$, and generally $p_n = p_1^{k^n}$. Also, a person of the nth generation will have 2^n ancestors all different; hence for homogeneity we have

$$2^n = N\{1 - p_1 - p_1^k - p_1^{k^2} - \ldots - p_1^{k^n}\}.$$

[Mr. WHITWORTH remarks that the least time is zero, the homogeneity being effected by $N-1$ of the persons instantaneously dying.]

6284. (By Professor COCHEZ.)—Soit $f(x) = 0$ une équation algébrique, α une quelconque de ses racines. Dans l'équation $\phi(y) = 0$ dont les racines ont avec x la relation $y = f'(x)$, (1) le terme du premier degré manque; (2) le terme tout connu est le dernier terme de l'équation aux carrés des differences.

Solution by G. TURRIFF, M.A.; W. J. C. SHARP, M.A.; *and others.*

If $f(x) = (a, b, c, d \ldots \mathfrak{X}x, 1)^n$, $f'(x) = n(a, b, c, d \ldots \mathfrak{X}x, 1)^{n-1} = y$, and the equation in y is the eliminant of

$$(a, b, c, d \ldots \mathfrak{X}x, 1)^{n-1} = \frac{y}{n}, \text{ and } (a, b, c, d \ldots)(x, 1)^n = 0;$$

hence the absolute term, the result of putting $y = 0$, is the discriminant, and

$$\frac{1}{n}\Sigma f'(a) = as_{n-1} + (n-1)\,bs_{n-2} + \&c. = 0,$$

which proves that the second term vanishes.

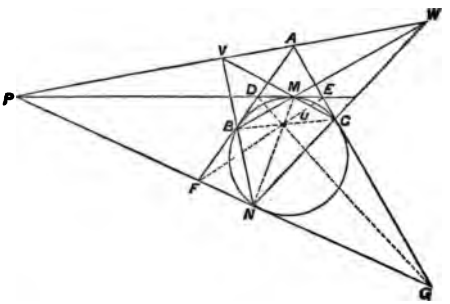
6179. (By R. KNOWLES, B.A., L.C.P.)—If a point P be taken on the asymptote to an hyperbola whose centre is C and focus S, and if PT be drawn a tangent to the curve, prove that $\angle TSP = SPC$.

Solution by A. L. SELBY, M.A.; E. RUTTER; *and others.*

The other tangent from P is the asymptote, and the line to the point of contact is therefore parallel to the asymptote; and since the tangents subtend equal angles at the focus, therefore $\angle PST$ = angle between SP and line parallel to asymptote = alternate $\angle SPC$.

6720. (By T. C. Simmons, M.A.)—AB, AC are two fixed tangents to a circle [or a conic]; DE, FG are two other tangents meeting AB in D, F, and AC in E, G; prove that the point of intersection of EF and DG lies in BC.

I. *Solution by* S. Constable, M.A.; Belle Easton; *and others.*

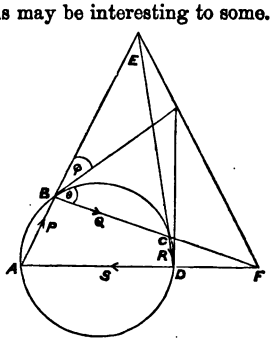


Let M, N be the points of contact of tangents DE, FG; let WV be the third diagonal of quadrilateral BNCM, and U the point of intersection of diagonals MN, BC. Then U is the pole of VW. Again, since the line joining the poles of two lines is the polar of their point of intersection, the line FE is the polar of V.

Similarly, DG is the polar of W; whence the point of intersection of FE and DG must be the pole of VW. But U is the pole of VW, therefore the lines DG, FE intersect in U, *i.e.*, on BC.

II. *Solution by* Professor Wolstenholme, M.A.

A proof of this from statical considerations may be interesting to some. Suppose forces P, Q, R, S to act along the chords AB, BC, CD, DA of a circle, and each to be inversely proportional to the chord along which it acts; let AB, CD meet in E, and AD, BC in F. Then the resultant of P, R acts through the point E, because both forces do so; and through the point F, because, the triangles FCD, FAB being similar, if p_1, p_2 be their altitudes, $\frac{p_1}{CD} = \frac{p_2}{BA}$, or $p_1 R = p_2 P$,



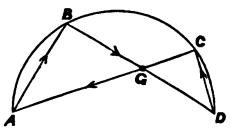
or the moments about F are equal and opposite. The resultant therefore acts along the straight line EF. Similarly, the resultant of Q, S also acts along EF, and therefore the resultant of the four forces also does so. But, if θ, ϕ be the angles which the tangent at B makes with BC, BE, we have

$$\frac{BC}{AB} = \frac{\sin\theta}{\sin\phi} = \frac{P}{Q};$$

whence the resultant of P, Q acts along the tangent at B. Similarly, the resultant of R, S acts along the tangent at C; and the resultant of P, Q, R, S must act through the point of intersection of the tangents at B, D.

This intersection is then a point on EF, and similarly, the tangents at A, C intersect on E, F. Thus, if a quadrilateral be formed by four tangents to a circle, a diagonal passes through two of the vertices of the quadrangle determined by the points of contact.

If we had taken the forces along AB, BD, DC, CA, and inversely proportional to these lengths, we should prove in the same way that the resultant of the four forces acts along EG, and also through the intersections of the tangents at B, C, and of the tangents at A, D.



The third case, where the resultant passes through F, G, and the intersections of the tangents at A, B, and of the tangents at C, D, will be proved by taking forces along AC, CB, BD, DA, inversely proportional to these lengths.

The general case, then, is, if we have four tangents (a, b, c, d) to a circle, the points of contact being A, B, C, D; the points (BC, AD), (AB, CD), tangents b, d, tangents a, c, lie on one straight line; the points (CA, BD), (BC, AD), tangents a, b, tangents c, d, lie on one straight line; and the points (AB, CD), (CA, BD), tangents b, c, tangents a, d, lie on one straight line. Of course the property is projective, and equally true for any conic.

6078. (By Edwyn Anthony, M.A.)—Prove that (1) the principal normal at any point of a helix passes through and is perpendicular to the axis of the cylinder on which the helix is traced; (2) the angle between the binormal and the axis equals the complement of the angle between the axis and the tangent.

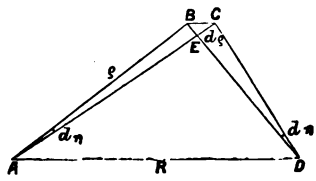
Solution by Belle Easton; D. Edwardes; *and others.*

1. The equations of the helix being $x = a\cos\theta$, $y = a\sin\theta$, $z = na\theta$, the equation of osculating plane is $n(\xi y - \eta x) + a(\zeta - z) = 0$, which is satisfied by $x = 0$, $y = 0$, $\zeta = z$, and $\frac{dz}{ds} = \sin\alpha$, $\therefore \frac{d^2z}{ds^2} = 0$, which proves (1).

2. If ϕ be the angle which the binormal makes with axis, and α the pitch of the screw, we have $\cos\phi = a\frac{d\theta}{ds} = \frac{1}{(1+n^2)^{\frac{1}{2}}} = \cos\alpha$.

6807. (By G. F. Walker, M.A.)—In any curve in which the difference between the radii of absolute and spherical curvature is constant, prove that the arcs of the loci of the centres of absolute and spherical curvature (measured between corresponding points) are equal.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.



Let us denote the elements of the curves described by the centre of absolute curvature and by the centre of spherical curvature, respectively, by ds and dS, the radius of absolute curvature by ρ, the radius of spherical curvature by R, and the angle of torsion by $d\eta$. Three consecutive elements of the curve having been taken, suppose the plane of the paper to be perpendicular to the second element at its middle point A, then B and C, the centres of absolute curvature, corresponding to the osculating planes containing the first and second elements, and the second and third, respectively, will be in the plane of the paper, as will also be the two consecutive *binormals*, or corresponding axes of osculating planes through B and C, intersecting in D, the centre of spherical curvature. The angles BAC and BDC will each be the angle of torsion $d\eta$, EC will be $d\rho$, and CD (or *ultimately* BD) will be $\frac{d\rho}{d\eta}$. As the consecutive centre of spherical curvature lies on CD, CD is a tangent to the locus of the centre of spherical curvature, and by a well-known theorem (WILLIAMSON'S *Diff. Calc*, 4th ed., p. 233), we have $d(\mathrm{BD}) = d\mathrm{S} - \mathrm{BE}$, or $d\mathrm{S} = \rho\, d\eta + d\frac{d\rho}{d\eta}$; but, as $\mathrm{R} - \rho = \text{const.}$, $(\mathrm{AB}^2 + \mathrm{BD}^2)^{\frac{1}{2}} - \rho = \text{const.}$, or $\left(\rho^2 + \frac{d\rho^2}{d\eta^2}\right)^{\frac{1}{2}} - \rho = \text{const.}$; or, differentiating, $\rho\, d\eta + d\frac{d}{d\eta} = (\rho^2 d\eta^2 + d\rho^2)^{\frac{1}{2}} = (\mathrm{BE}^2 + \mathrm{EC}^2)^{\frac{1}{2}} = \mathrm{BC} = ds$; therefore $d\mathrm{S} = ds$, or $\mathrm{S}_1 - \mathrm{S}_0 = s_1 - s_0$.

More generally, if R and ρ be connected by the relation $\mathrm{R} - k\rho = c$, where k and c are both constants, including the case in which c is zero, and, consequently, the ratio of R to ρ is given, it follows nearly as above that $\mathrm{S}_1 - \mathrm{S}_0 = k(s_1 - s_0)$.

[The radius of curvature of the edge $\left(\frac{d\mathrm{S}}{d\mathrm{T}}\right) = \frac{\mathrm{R}\,d\mathrm{R}}{d\rho}$, and $d\mathrm{S} = \mathrm{R}\,d\mathrm{T}$; therefore, since $d\mathrm{R} = d\rho$, we see that $ds = \mathrm{R}\,d\mathrm{T} = ds\theta$, and $\mathrm{S} = s$ between corresponding points.]

6398. (By R. E. RILEY, B.A.)—Prove that if $(a+b+c)^3 = a^3 + b^3 + c^3$, then $(a+b+c)^{2n+1} = a^{2n+1} + b^{2n+1} + c^{2n+1}$.

Solution by E. W. SYMONS, M.A.; G. TURRIFF, M.A.; *and others*.

We have $(a+b+c)^3 \equiv a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b) = a^3 + b^3 + c^3$,
therefore $(b+c)(c+a)(a+b) = 0$;
and, putting any one of these factors $= 0$, the required result follows.

6270. (By J. L. McKenzie, B.A.)—Express in terms of the coefficients of two quadratic equations, the condition that one root of the first should have a given ratio to one root of the second.

Solution by W. J. Constable, B.A.; J. O'Regan; *and others.*

Let the equations be $a_1x^2+b_1x+c_1=0$, $a_2x^2+b_2x+c_2=0$, and the ratio $m:1$; then, eliminating x from

$$a_1x^2+b_1x+c_1=0, \quad a_2m^2x^2+b_2mx+c_2=0,$$

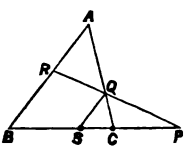
the required condition is

$$(b_1c_2-b_2c_1m)(b_1a_2m^2-a_1b_2m)=(c_1a_2m^2-a_1c_2)^2.$$

6092. (By J. Young, B.A.)—Through a point in the base of a triangle produced draw a straight line cutting the sides so that the rectangle contained by the segment of one side towards the base and the segment of the other towards the vertex of the triangle shall be a maximum.

Solution by G. Heppel, M.A.; Christine Ladd; *and others.*

Drawing QS parallel to AB, let PC $= h$, PB $= k$, PS $= x$, then $k-h=a$,



$$QC=\frac{b}{a}(x-h), \quad AR=c-\frac{kc}{ax}(x-h);$$

$$AR.QC=\frac{bc}{a}\left\{x-h-\frac{k}{a}\left(x-2h+\frac{h^2}{x}\right)\right\}$$

$$=\text{a maximum when } 1-\frac{k}{a}+\frac{kh^2}{ax^2}=0,$$

that is, when $(k-a)x^2=kh^2$, or $x^2=kh$,

so that then PS is a mean proportional between PB and PC.

Exactly the same result is obtained if the rectangle be AQ . BR.

6800. (By the Editor.)—Prove that, if

$$\frac{ayz}{y^2+z^2}=\frac{bzx}{z^2+x^2}=\frac{cxy}{x^2+y^2}=1, \text{ then } a^2+b^2+c^2=abc+4.$$

Solution by Arthur Cohen, Q.C., M.P.; E. Rutter; *and others.*

Let $l=\frac{y}{z}$, $m=\frac{z}{x}$, $n=\frac{x}{y}$; then we have $lmn=1$,

$$a = l+mn, \quad b = m+nl, \quad c = n+lm, \quad \Sigma a^2 = \Sigma l^2 + \Sigma m^2n^2 + 6;$$

$$abc = \frac{1}{lmn}\{(lmn+l^2)(lmn+m^2)(lmn+n^2)\}.$$

$$= (1+l^2)(1+m^2)(1+n^2) = 1+\Sigma l^2+\Sigma m^2n^2+1 = \Sigma a^2-4.$$

[Otherwise:—$(a^2+b^2+c^2)\,x^2y^2z^2 = x^2(y^2+z^2)^2 + y^2(z^2+x^2)^2 + z^2(x^2+y^2)^2$ $\equiv (x^2+y^2)(y^2+z^2)(z^2+x^2) + 4x^2y^2z^2 = abc\,x^2y^2z^2 + 4x^2y^2z^2$; therefore &c.]

NOTE ON QUESTION 6800; *by* Professor CAYLEY, F.R.S.

The identity given by the solution is a very interesting one. Instead of a, b, c, writing $(a, b, c) + d$, we have $4d^3 - d(a^2+b^2+c^2) + abc = 0$, satisfied by $a : b : c : d = x(y^2+z^2) : y(z^2+x^2) : z(x^2+y^2) : xyz$; or, considering (a, b, c, d) as the coordinates of a point in space, and (x, y, z) as the coordinates of a point in a plane, we have thus a correspondence between the points of the cubic surface $4d^3 - d(a^2+b^2+c^2) + abc = 0$, and the points of the plane. To a given system of values of (x, y, z) there corresponds, it is clear, a single system of values of (a, b, c, d); and it may be shown without difficulty that to a given system of values of (a, b, c, d) satisfying the equation of the surface there correspond two systems of values of (x, y, z); the plane and cubic surface have thus a (1, 2) correspondence with each other.

6432. (By J. HAMMOND, M.A.)—Prove that PG $= 2\rho$ if PG be the normal measured from the curve to the axis of x, and ρ the radius of curvature of the curve

$$x = \frac{c}{2^{\frac{1}{2}}}\int \frac{d\theta}{(1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}} - c\sqrt{2}\int (1-\tfrac{1}{2}\sin^2\theta)^{\frac{1}{2}}\,d\theta, \quad y = c\cos\theta.$$

Solution by C. MORGAN, B.A.; D. EDWARDES; *and others.*

The property PG $= 2\rho$ leads to the equation

$$py\frac{dp}{dy} + 2p^2 + 2 = 0, \quad \left(p \equiv \frac{dy}{dx}\right), \quad \text{whence} \quad 1+p^2 = \frac{c^4}{y^4}.$$

Putting $y = c\cos\theta$, this gives $\dfrac{dx}{d\theta} = \dfrac{c\cos^2\theta}{(1+\cos^2\theta)}$,

therefore $x = c\sqrt{2}\int (1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}\,d\theta - \dfrac{c}{2^{\frac{1}{2}}}\int \dfrac{d\theta}{(1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}}$.

6289. (By G. J. Griffiths, M.A.)—A uniform heavy string of length $2a$ is placed on a smooth cardioid $r = a(1+\cos\theta)$, whose axis is horizontal; one end of the string being at the apse; and the string is allowed to run off the curve; prove that its velocity v when just leaving the curve is given by the equation $v^2 = \frac{1}{10}ga(52-3^{\frac{5}{2}})$.

Solution by D. Edwardes; J. O'Regan; *and others.*

If θ be the vectorial angle of a point on the curve, and ϕ the inclination of the tangent at that point to the vertical, $\phi = \frac{3}{2}\theta$. Let z be the length of string which is vertical at the time t. Then, equating the rate of increase of momentum to the impressed forces, we have

$$2av\frac{dv}{dz} = gz + g\int_0^{2a-z}\cos\phi\, ds;$$

the mass of a unit length of string being taken as the unit of mass, and s is measured from the apse.

Now $s = 4a\sin\frac{1}{2}\theta$, $\therefore \int\cos\phi\, ds = 4a\sin\frac{1}{2}\theta\cos^3\frac{1}{2}\theta = s\left(1-\frac{s^2}{16a^2}\right)^{\frac{3}{2}}$.

Hence, integrating, and putting $2a-z = s'$, we get

$$av^2 = \tfrac{1}{2}gz^2 - g\int s'\left(1-\frac{s'^2}{16a^2}\right)^{\frac{3}{2}}ds' + C = \tfrac{1}{2}gz^2 + \tfrac{1}{5}a^2g\left(1-\frac{s'^2}{16a^2}\right)^{\frac{5}{2}} + C,$$

and $v = 0$ when $s' = 2a$, therefore $C = -\frac{3^{\frac{1}{2}}}{10}a^2g$. Hence, &c., putting $s' = 0$.

6649. (By J. O'Regan.)—If a quadrilateral be circumscribed to a circle and a fifth variable tangent be drawn, the rectangles under perpendiculars on it from each pair of opposite angles are in a constant ratio.

Solution by G. Heppel, M.A.; Rev. T. R. Terry, F.R.A.S.; *and others.*

Taking the centre of the circle as origin, and representing the vectorial angles of the points of contact by α, β, γ, δ, the equations to the four tangents are $x\cos\alpha + y\sin\alpha - a = 0$, &c.

The coordinates of the intersection of the first two are

$$\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)},\quad \frac{a\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

The length of the perpendicular from this point on the variable tangent

$$x\cos\theta + y\sin\theta - a = 0, \text{ is } 2a\sin\tfrac{1}{2}(\theta-\alpha)\sin\tfrac{1}{2}(\beta-\theta)\sec\tfrac{1}{2}(\alpha-\beta);$$

hence the ratio of rectangles under the perpendiculars is

$$\frac{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\delta-\alpha)}{\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\gamma-\delta)}.$$

6765. (By Prince CAMILLE DE POLIGNAC.)—Find the condition for the convergence of the continued fraction $A+\cfrac{p}{q+}\cfrac{p}{q+}\cfrac{p}{q+}$ &c.

Solution by the PROPOSER.

Starting with $x_n=\dfrac{Ax_{n-1}+B}{A'x_{n-1}+B'}$, and transforming this expression as follows,

$$A'x_n=\frac{A(A'x_{n-1}+B')+BA'-AB'}{A'x_{n-1}+B'}=A+\frac{BA'-AB'}{A'x_{n-1}+B'},$$

$$A'x_n+B'=A+B'+\frac{BA'-AB'}{A'x_{n-1}+B'};$$

putting $BA'-AB'=p$, $A+B'=q$, we have

$$A'x_n+B'=q+\frac{p}{A'x_{n-1}+B'},\quad \text{therefore } A'x_n=A+\cfrac{p}{q+}\cfrac{p}{q+}\ldots\frac{p}{A'x_0+B'},$$

where p occurs n times.

This expansion gives, to a factor, the n^{th} power of the linear substitution we started with. Now we know, from the solution to Quest. 6735,* where these powers have been represented as homographic points on a line, that they converge towards the foci of the system when real.

Hence the condition of convergence is that the roots of the quadratic $A'x^2+(B'-A)x-B=0$ should be real, that is to say, that

$$(B'-A)^2+4BA'\geqq 0 \text{ or } (B'+A)^2+4(BA'-AB')\geqq 0, \text{ or } q^2+4p\geqq 0.$$

6783. (By R. KNOWLES, B.A., L.C.P.)—Prove that (1) if the axes be rectangular, the equation to the locus of the vertices of all parabolas whose chords of contact cut off a triangle of constant area ($\frac{1}{2}a^2$), is $(x^7y^{-1})^{\frac{1}{3}}+(y^7x^{-1})^{\frac{1}{3}}=a^{\frac{1}{3}}$; (2) that of the foci is $(x^2+y^2)^2=a^2xy$; (3) the chords of contact always touch at their middle points a rectangular hyperbola, to which the axes are asymptotes.

Solution by W. B. GROVE, B.A.; CHRISTINE LADD; *and others.*

Let the equation to the parabola be $\left(\frac{x}{h}\right)^{\frac{1}{2}}+\left(\frac{y}{k}\right)^{\frac{1}{2}}=1$, where $hk=a^2$; then the chord of contact and perpendicular thereon from the origin are

$$\frac{x}{h}+\frac{y}{k}=1,\ ky=hx \quad \ldots\ldots\ldots\ldots\ldots\ldots(1, 2);$$

and the coordinates of the intersection of (1), (2), that is, the focus S, are $\frac{hk^2}{h^2+k^2}$, $\frac{h^2k}{h^2+k^2}$; therefore the locus of S is $(x^2+y^2)^2=a^2xy$.

* See pp. 93 and 94 of this volume.

Again, the equation to the line through S, making with the axis of x an angle equal to that which the chord of contact makes with the same, and the coordinates of this line's intersection with the curve, *i.e.*, the *vertex*, are

$$y - \frac{h^2k}{h^2+k} = \frac{k}{h}\left(x - \frac{hk^2}{h^2-k}\right), \quad \frac{hk^2}{(h+k)}, \quad \frac{h^2k}{(h^2+k^2)^2};$$

hence the locus of the vertex is $\left(\frac{x}{y}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{1}{2}} = \left(\frac{a^2}{xy}\right)^{\frac{1}{2}}$.

Lastly, the envelop of (1), subject to the condition $hk = a^2$, is $4xy = a^2$, and the coordinates of the point of contact are $\frac{1}{2}h$, $\frac{1}{2}k$; hence the chords of contact always touch at their middle points, &c.

6572. (See p. 62). II. *Solution by* J. Young, B.A.; M. Baker, M.A.; *and others.*

In the diagram to the following Solution, let the inner circle touch AB in F, and let the circle through AOB cut OF produced in G; then, since OD = AD − OF, we have $OD^2 = AD^2 + OF^2 - AB \cdot OF = OF^2 + FD^2$;

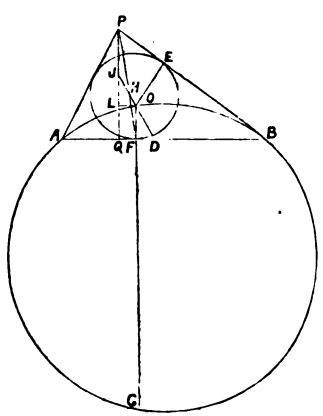
therefore $AB \cdot OF = AD^2 - FD^2 = AF \cdot FB = OF \cdot FG$;

hence AB = FG = the radius of the circle touching AB and the lines PA, PB produced, because the centre of this circle lies on the circle AOB, and the distance from B of its point of contact with AB is equal to AF; therefore AB . PE = area PAB = $\frac{1}{2}$PQ . AB, hence PE = $\frac{1}{2}$PQ; therefore, &c.

The proof is similar for the external contact.

III. *Solution by* D. Edwardes; Prof. Matz, M.A.; *and others.*

It may be shown that the line DO, which joins the centre of the inscribed circle to the middle point of the base, bisects in H the line PF joining the vertex to the point where the inscribed circle touches the base. Writing then for shortness p, r, c, t for PQ, OF, AB, PE respectively, putting J at the intersection of OD and PQ, and drawing OL at right angles to PQ, we have PJ = r, and



$$OP^2 = PJ^2 + OJ^2 + 2PJ \cdot JL,$$

or $t^2 = OJ^2 + 2r(p - 2r)$..... (1).

By similar triangles,

$$\frac{p-2r}{OJ} = \frac{2r}{c-2r} \text{ (since } OD = \tfrac{1}{2}c - r).$$

But area of PAB = $\frac{1}{2}pc$

and also $= r(c+t)$ ... (2),

or $$\frac{p-2r}{2r} = \frac{t}{c} = \frac{t-p+2r}{c-2r};$$

therefore $OJ = t - p + 2r$;

and therefore, by (1),

$$t^2 = (t-p+2r)^2 + 2r(p-2r), \text{ whence } p = 2t \text{ ..... (3).}$$

Hence equations (2) and (3) give $\frac{1}{r} - \frac{2}{p} =$ constant, which (since $\frac{2r}{p} = \tan \frac{1}{2}P$), is equivalent to $\tan \frac{1}{2}A + \tan \frac{1}{2}B = 1$.

[That D, O, H are in a straight line follows from the vanishing of the determinant $\begin{vmatrix} \frac{\Delta}{a}, & \frac{\Delta}{b}, & 0 \\ r, & r, & r \\ r\cos\frac{1}{2}B, & r\cos^2\frac{1}{2}A, & \frac{rs}{c} \end{vmatrix}$, whereof the rows are the trilinear coordinates of these three points respectively.]

6834. (By the Editor.)—Trace the locus of a point whose distance from a given line $y + c = 0$ is equal to the sum (including, throughout, *difference*) of its distances from two given points (a, b), $(-a, -b)$, showing that (1) its general equation is

$$[(y+c)^2 + 4(ax+by)]^2 = 4(y+c)^2[(x+a)^2 + (y+b)^2];$$

(2) when $b = 0$, so that the curve is the locus of the vertex of a triangle on a given base, having the sum of its two sides equal to the sum of the perpendicular from the vertex on the base and a given line (c), the equation is $(y+c)^4 + 16a^2x^2 = 4(y+c)^2(a^2+x^2+y^2)$; (3) when $c > 2a$, the curve (2) consists of a loop and four infinite branches that cross at the intersection of the y-axis with the given line, and have an asymptote parallel to that line at a distance $2a$ on each side therefrom; (4) analogous but varying forms subsist when $2a > c > a\sqrt{3}$ and $c < a\sqrt{3}$, with a conjugate point on the y-axis when $c = a\sqrt{3}$; (5) when $c = 2a$, so that, in the triangle-locus (2), the sum of the sides is equal to the sum of the base and perpendicular, the equation is $4(a^2 - x^2)(y + 4a) = (3y + 8a)y^2$, and then two of the infinite branches degenerate into the base of the triangle, and the other two unite continuously with the loop and cut the y-axis at an angle $48° 12' = \cos^{-1}\frac{1}{3}$; (6) the curve (5) is also the locus of the intersection of tangents drawn from the ends of a diameter of a circle to a circle that touches the given circle and its diameter; (7) the areas of the loop of the curve (5), of the part cut off by the base, and of the space between the infinite branches and the asymptote, are, if we put $\sin^2\beta = \frac{5}{8}$,

$$\frac{2}{3^{\frac{3}{2}}}(7\sqrt{15} - 16\beta)a^2, \quad \frac{16}{27}(9 - \pi\sqrt{3})a^2, \quad \frac{2}{3^{\frac{3}{2}}}(7\sqrt{15} + 16\beta)a^2;$$

(8) the locus of the centre of the touching circle in (6) is a parabola; and (9) the sum of the tangents of the angles subtended at each end of the given diameter in (6) by lines drawn from the other end to the centre of the touching circle, is unity. (See solution of Quest. 6572.)

Solution by D. Edwardes; G. M. Reeves, M.A.; *and others.*

1. We have $[(x+a)^2 + (y+b)^2]^{\frac{1}{2}} \pm [(x-a)^2 + (y-b)^2]^{\frac{1}{2}} = y + c$,
and, since $[(x+a)^2 + (y+b)^2] - [(x-a)^2 + (y-b)^2] \equiv 4(ax+by)$,

therefore $[(x+a)^2+(y+b)^2]^{\frac{1}{2}} \mp [(x-a)^2+(y-b)^2]^{\frac{1}{2}} = 4(ax+by)(y+c)^{-1}$, whence, by adding and squaring, we get the result stated.

2, 3, 4. Putting $b = 0$, we have the result stated, which is equivalent to $$4x^2[(y+c)^2-4a^2] = (y+c)^2[(y+c)^2-4(y^2+a^2)],$$ or $$4x^2(2a-c-y)(2a+c+y) = 3(y+c)^2(y-a_1)(y-a_2),$$ where $$a_1, a_2 \equiv \tfrac{1}{3}[c \pm 2(c^2-3a^2)^{\frac{1}{2}}].$$

By the usual method, the asymptotes are found to be $y+c = \pm 2a$, as stated in the Question. When $y = 0$, $x = \pm \frac{1}{2}c$, and the curve cuts the x-axis at $\tan^{-1} 2$. When $y = -c$, $x = 0$, and the tangents are $(c^2+a^2)y^2-4a^2x^2=0$. a_2 is + or −, according as $c \lesseqgtr 2a$, also $a_2 \gtreqless 2a-c$, according as $c \gtreqless 2a$. When $c < 2a$, then, while y is positive and $< a_2$, x is $\pm$. When $y = a_2$, the curve cuts the y-axis at right angles. Also, c being less than $2a$, $a_2 < 2a-c$; hence, if $a_1 > y > a_2$, x is $\sqrt{-1}$. At $y = a_1$, the curve cuts the y-axis at right angles. When $2a-c > y > a_1$, x is $\pm$; and when $y = 2a-c$, $x = \infty$. The several cases are exhibited in the diagrams. As c increases, the asymptotes continually descend. When $c = a\sqrt{3}$, there is a conjugate point on the y-axis, where $y = \frac{1}{3}c$, and above the asymptote. Figs. 1, 2 are respectively drawn for $c = a$, $1\frac{1}{2}a$ nearly.

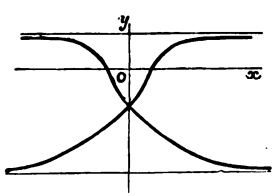


Fig. 1. $c < a\sqrt{3}$.

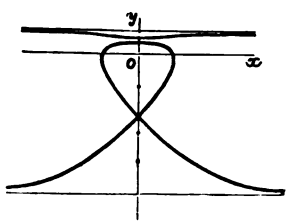


Fig. 2. $2a > c > a\sqrt{3}$.

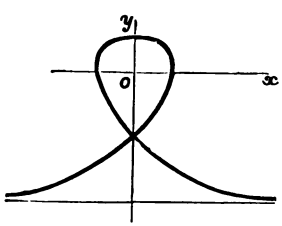


Fig. 3. $c = 2a$.

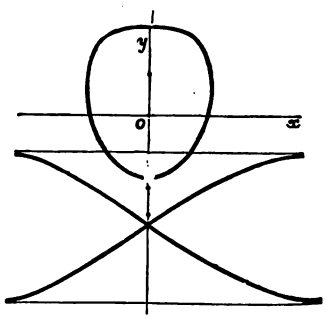


Fig. 4. $c > 2a$.

5. When $c = 2a$, the equation becomes $4x^2(y+4a) = (y+2a)^2(4a-3y)$ or $4(a^2-x^2)(y+4a) = (3y+8a)y^2$.

6. In the Fig. to Quest. 6572 (see pp. 62, 11′), let PQ = y, DQ = x, $\tan\frac{1}{2}A \tan\frac{1}{2}B = z$, then, using the property $\tan\frac{1}{2}A + \tan\frac{1}{2}B = 1$, we get

$$z = \frac{4(a^2-x^2)}{y^2} - 2, \quad \frac{2ay}{a^2-x^2} = \frac{2-2z}{2z+z^2},$$

whence, eliminating z, we have the equation in (5).

7. [Our correspondents all find the areas from the equation in (5), but they may be more simply obtained by removing the origin to the vertex or top of the loop (Fig. 3), and taking the central line as positive x-axis, when the equation becomes

$$4(16a-3x)y^2 = x(10a-3x)^2;$$

whence, putting $3x = 16a\sin^2\theta$, the area of any part of the curve is

$$\Sigma = \int y\,dx = \int y\left(\frac{dx}{d\theta}\right)d\theta = \frac{16a^2}{3\sqrt{3}}\int(-1+3\cos 2\theta - 2\cos 4\theta)\,d\theta$$
$$= \tfrac{8}{9}\sqrt{3}\,(-2\theta+3\sin 2\theta-\sin 4\theta)\,a^2.$$

The auxiliary angle θ is the angle subtended at the lower (or asymptotic) end of the vertical axis by the horizontal y-ordinate to a circle on the whole axis as diameter; and the angle β is the value of θ at the node.

The areas in the question, in order, are obtained by taking the double of Σ between the θ-limits $(0, \beta)$, $(0, \frac{1}{3}\pi)$, $(\beta, \frac{1}{2}\pi)$.]

8. The centre O of the variable circle (see Figures to solution of Question 6572, pp. 62, 110) is equidistant from D and from the tangent to the semicircle parallel to AB; hence the locus of O is a parabola.

[The general curve may be readily extended so as to give the locus of a point whose distance from a fixed straight line bears a given ratio to the sum or difference of its distances from two fixed points.]

6861. (By Professor Genese, M.A.)—A, B, C, D are fixed points. A circle is drawn through AB; then two can be drawn through CD to touch it; prove that the locus of the points of contact is a bicircular quartic.

Solution by Prof. Townsend, F.R.S.; Prof. Nash, M.A; *and others.*

More generally, A, B, C, D, E, F being fixed points in a common plane, if a conic through ABEF touch a conic through CDEF, the locus of the points of contact is a binodal quartic of which E and F are the nodal points. For, if U and V be the two conics passing through all the points except B and A respectively; then, since six conics may be described through ABEF, three touching U at E, at F, and at A respectively, and three touching V at E, at F, and at B respectively, the two points A and B, and for a similar reason the two C and D, are single, and the two E and F are double points of the locus in question; the order of which is seen to be fourth, from the consideration that the two lines AB and CD can meet it again, the former only at the two points P and Q of its contact with two

of the system of conics through CDEF, and the latter only at the two points R and S of its contact with two of the system of conics through ABEF. When the two points E and F, common to the two systems of conics, are the two I and J in the plane of the points, the property is manifestly that of the question.

When the six points A, B, C, D, E, F lie on a conic in the plane, the locus in question breaks up into the containing conic, and another intersecting it at E and F, and also at the points of contact M and N of the tangents to it from the intersection of the lines AB and CD; the pairs of tangents to the two conics at M and N dividing harmonically the angles EMF and ENF at which M and N are subtended by EF; and the two conics themselves becoming two circles intersecting at right angles when E and F coincide with I and J, as in the question.

[Inverting with respect to C, we have to find the locus of the points of contact of tangents from a fixed point D to a circle passing through two fixed points. If a be constant and α variable, the equations of circle and polar of (X, Y) are

$$x^2+y^2-a^2 = 2\alpha y, \quad x\mathrm{X}+y\mathrm{Y}-a^2 = \alpha(y+\mathrm{Y}) \ldots\ldots\ldots\ldots (1, 2);$$

and locus of intersection of (1), (2) is

$$(x^2+y^2-a^2)(y+\mathrm{Y}) = 2y(x\mathrm{X}+y\mathrm{Y}-a^2),$$

a circular cubic, the inverse of which is a bicircular quartic, passing through the four points A, B, C, D.]

NOTE ON QUESTION 6816. *By the* EDITOR.

In reference to the solution given on pp. 101, 102, Mr. WHITWORTH states that he does "not quite see what Dr. MACFARLANE means, but his solution, I think, must be wrong because 'least time' cannot be a function of 'averages' or 'probabilities.' To calculate the least time in which a result can be brought about, we have nothing whatever to do with the *chances* of the causes which bring about that result, but each must be assumed to happen in the way most favourable for the result. I am quite serious in offering, as the only true solution of the question, that which has been bracketed as a remark at the foot of the solution."

To this Dr. MACFARLANE replies, that he "cannot altogether agree with the principle which Mr. WHITWORTH lays down; for, may we not consider all the causes but one to act in an average manner, and investigate what follows from supposing the remaining cause to act in the most favourable way? It is assumed, in my solution, that the colony has an average experience in every respect, excepting that they adopt the law of marriage which is most favourable for producing homogeneity. The minimum refers to this last variable. Mr. WHITWORTH's may be the absolute minimum, but mine is a relative, and I think the relevant, minimum. By the term homogeneous is meant, descended from a common group of ancestors, with whatever that fact may imply. If $N(1-p_1-p_1^k-\&c.)$ is not an exact power of 2, the number next above, which is a power of 2, is to be taken."

6092 (See p. 106). II. *Solution by the* PROPOSER.

The triangle CQR (see Fig. to former solution) is a maximum when the vertex S of the parallelogram ARQ lies on BC (MULCAHY'S *Geometry*, p. 88); and in this position of S, PC . PS = PQ : PR = PS : PB, or PS is a mean proportional between PC and PB. Draw RT perpendicular to AC: then the triangle CQR = ½CQ . RT, and RT is constantly proportional to AR, therefore, &c. Since CQ . AR : AQ . BR = CP : PB, the rectangles CQ . AR and AQ . BR are a maximum at the same time.

6882. (By BELLE EASTON.)—Through a point P, between two lines AB, AC given in position, draw a line such that the rectangle under the parts thereof between the point and those lines may be a minimum.

Solution by ARTHUR COHEN, Q.C., M.P.; J. O'REGAN; *and others.*

Let qPr be the required line, so that $Pq \cdot Pr$ is a minimum; and let $q'Pr'$ be a contiguous line; then it is evident that ultimately $Pq \cdot Pr$ must equal $Pq' \cdot Pr'$; therefore the points q, q', r, r' must lie on a circle, and AB, AC must be tangents to the circle at q and r respectively; therefore $Aq = Ar$. Draw then AD bisecting the angle A, and Pqr perpendicular on it; then Pqr is the required line.

PROOF THAT AN EQUATION MUST HAVE AT LEAST n ROOTS.
By J. HAMMOND, M.A.

The roots of the equation $x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = 0$, if any such exist, can only be functions of n and of the coefficients; *i.e.*, they must be of the form $\phi(n, a, b, c, \ldots l)$. The roots of any special equation will be the results of substituting special values for $a, b, c, \ldots l$ in each of the functions ϕ. Now, if there were only m of these functions, we could not have *more* than m special values of them, all distinct from one another, though we might have *less* than m *distinct* values of the functions in some special cases. Or, the number of *distinct* roots of a special equation *cannot be more, though it may be less*, than the number of roots of the general equation. Now, in the special case $x^n = 1$, we know of n distinct roots; therefore n cannot be greater, though it may be less, than the number of roots of the general equation of the n^{th} degree; or, the general equation of the n^{th} degree cannot have less than n roots.

6778. (By J. J. WALKER, M.A.)—Show that the work done (in *C.G.'s*) in raising the piston of a suction pump, so as to elevate the water *b cms* in

the suction-tube (of section a) from a point a *cms* below the bottom of the working-barrel, is equal to

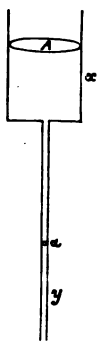
$$a\left\{\tfrac{1}{2}b^2 - a\mathrm{H}\log_e\left[\mathrm{H}+(\mathrm{H}-b)\right] + \mathrm{H}ab \div (\mathrm{H}-b)\right\},$$

H being the height of the water-barometer; and explain the significance of the separate terms.

Solution by G. F. Walker, M.A.; G. M. Reeves, M.A.; *and others.*

Work in *C.G.'s* is $\int \frac{\mathrm{A}dx\,(\Pi - p)}{g\rho}$, where A is area of piston and x the height through which it is raised.

Let y be height to which water rises.



Then $p + g\rho y = \Pi$, and $p\left[\mathrm{A}x + a(a-y)\right] = \pi a a$;

$$\mathrm{A}x + a(a-y) = \frac{a a \mathrm{H}}{\mathrm{H}-y},$$

$$\mathrm{A}dx = \left[a + \frac{a a \mathrm{H}}{(\mathrm{H}-y)^2}\right] dy;$$

$$\text{work} = \int_0^b y\left[a + \frac{a a \mathrm{H}}{(\mathrm{H}-y)^2}\right] dy,$$

piston raised slowly,

$$= a\left[\tfrac{1}{2}b^2 + a\mathrm{H}^2\int_0^b \frac{dy}{(\mathrm{H}-y)^2} - a\mathrm{H}\int_0^b \frac{dy}{\mathrm{H}-y}\right]$$

$$= a\left[\tfrac{1}{2}b^2 + \frac{ab\mathrm{H}}{\mathrm{H}-b} - a\mathrm{H}\log\frac{\mathrm{H}}{\mathrm{H}-b}\right].$$

[The first term is the work done in raising a column of water of height b, irrespective of atmospheric pressure; the second is the work done against external pressure; the third, that done by internal pressure, in the process of expanding the column of air occupying the suction tube from pressure H to pressure $\mathrm{H}-b$.]

6859. (By Professor Simon Newcomb, M.A.)—Prove that

$$\log\left(1 - \frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2 + \tfrac{1}{2}\eta^4 - \tfrac{1}{3}\eta^6 + \ldots - 2\eta\cos x$$

$$-\tfrac{1}{2}\cdot 2\eta^2\cos 2x - \tfrac{1}{3}\cdot 2\eta^3\cos 3x - \ldots = \Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i} - \Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

Solution by Dr. Macfarlane, F.R.S.E.

In the *Mathematical Visitor* for 1880, I proposed an expansion, the proof of which involves that of Professor Newcomb. It is

$$\log\left[a^2 + b^2 + 2ab\cos(\alpha-\beta)\right]^{\frac{1}{2}} = \log a + \frac{b}{a}\cos(\alpha-\beta) - \tfrac{1}{2}\frac{b^2}{a^2}\cos 2(\alpha-\beta) + \text{etc.}$$

I was led to this expansion by the following considerations:—

Let a denote a vector of length a and angle α, and b a vector of length b and angle β, and so on, the vectors being restricted to lying in one plane; then, as the combinations of such quantities are subject to all the laws of the ordinary algebra, all the known theorems are true when we substitute such directed quantities for the ordinary linear quantities. Thus, for the particular example, we have

$$\log(\mathrm{a}+\mathrm{b}) = \log \mathrm{a} + \log\left(1+\frac{\mathrm{b}}{\mathrm{a}}\right). \quad \text{Now, } \log \mathrm{a} = \log a + i\log\alpha,$$

$$\log\left(1+\frac{\mathrm{b}}{\mathrm{a}}\right) = \frac{\mathrm{b}}{\mathrm{a}} - \tfrac{1}{2}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^2 + \tfrac{1}{3}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^3 - \ldots$$

$$= \frac{b}{a}\cos(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2 \cos 2(\beta-\alpha) + \ldots$$

$$+ i\left\{\frac{b}{a}\sin(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2 \sin 2(\beta-\alpha) + \ldots\right\}.$$

Also $\log(\mathrm{a}+\mathrm{b}) = \log\left\{[a^2+b^2+2ab\cos(\beta-\alpha)]^{\frac{1}{2}}.\tan^{-1}\dfrac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right\}$,

where the dot separates the tensor and versor parts,

$$= \tfrac{1}{2}\log[a^2+b^2+2ab\cos(\beta-\alpha)] + i\log\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta};$$

therefore $\log[a^2+b^2+2ab\cos(\beta-\alpha)]$

$$= 2\log a + 2\left\{\frac{b}{a}\cos(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2\cos 2(\beta-\alpha) + \text{etc.}\right\} \quad \ldots\ldots(1),$$

and $\log\tan^{-1}\dfrac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}$

$$= \log a + \frac{b}{a}\sin(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2\sin 2(\beta-\alpha) + \ldots \quad \ldots\ldots(2).$$

Let $a^2+b^2=1$, $ab = \dfrac{-\eta}{1+\eta^2}$, and $\beta-\alpha = x$, then equation (1) becomes the expansion required by Prof. Newcomb. Equation (2) gives the complementary expansion.

The expansion may also be proved as follows:—

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\log(1+\eta^2) + \log[1-\eta(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}}+\eta^2),$$

$$= -\log(1+\eta^2) + \log(1-\eta\epsilon^{x\sqrt{-1}}) + \log(1-\eta\epsilon^{-x\sqrt{-1}}),$$

$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \text{etc.} - \eta(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}}) - \tfrac{1}{2}\eta^2(\epsilon^{2x\sqrt{-1}}+\epsilon^{-2x\sqrt{-1}}) - \text{etc.}$$

$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \text{etc.} - 2(\eta\cos x + \tfrac{1}{2}\eta^2\cos 2x + \text{etc.}).$$

6826. (See p. 97). II. *Solution by* W. W. Taylor, M.A.

The former solution of this problem contains the following curious error:—It is at first correctly proved that every measure of D_{n+1} and D_n is a

measure of nD_{n-1}, and also that every measure of D_{n+1} and n is a measure of D_n. The inference from these two facts is not, as stated, that the G. C. M. of D_{n+1} and D_n must be that of D_n and D_{n-1}, but that the former G. C. M. can only differ from the latter by being multiplied by factors of the latter. Hence, instead of 2 being the G. C. M., we can only infer that the G. C. M. is some power of 2. It is easily seen, by calculating the first seven values of D, that 2 is not the G. C. M. To complete the investigation, assume that 2^m is the G. C. M. of D_{4m-2} and D_{4m-1}; and let

$$D_{4m-2} = A2^m, \quad D_{4m-1} = B2^m,$$

where one of the two quantities A, B is odd; then, writing M for *some multiple of*, we get

$$D_{4m} = (8m-1)D_{4m-1} - (4m-1)D_{4m-2} = M2^{m+2} + (A-B)2^m,$$

$$D_{4m+1} = (8m+1)D_{4m} - 4mD_{4m-1} = M2^{m+2} + (A-B)2^m,$$

$$D_{4m+2} = (8m+3)D_{4m+1} - (4m+1)D_{4m} = M2^{m+2} + 2(A-B)2^m$$
$$= \{M2 + (A-B)\}2^{m+1};$$

$$D_{4m+3} = (8m+5)D_{4m+2} - (4m+2)D_{4m+1} = M2^{m+2} + 8(A-B)2^m = M2^{m+2}.$$

Thus, if D_{4m-2}, D_{4m-1} are one an odd and the other an even multiple of 2^m, D_{4m}, D_{4m+1} are both odd multiples of 2^m; D_{4m+2} is an odd multiple of 2^{m+1}, and D_{4m+3} is a multiple of 2^{m+2}; and, for one value of m, D_{4m-2} is an odd multiple of 2^m, and D_{4m-1} is an even multiple of 2^m, namely, when $m = 1$; therefore D_{4m}, D_{4m+1}, D_{4m+2}, D_{4m+3} always have values of the form found above. Therefore the G. C. M. of D_{4m} and D_{4m+1} is 2^m, of D_{4m+1} and D_{4m+2} is 2^m, of D_{4m+2} and D_{4m+3} is 2^{m+1}, of D_{4m+3} and D_{4m+4} is 2^{m+1}; or, generally, of D_n and D_{n+1}, the G. C. M. is 2^k, where k is the greatest integer in $\frac{1}{4}(n+2)$.

6770. (By Professor WOLSTENHOLME, M.A.)—A parallelogram of minimum perimeter is inscribed in a given ellipse; prove that a conic can be found which cuts the ellipse at right angles at each of the angular points of the parallelogram, and that this conic passes through four fixed points (two real and two impossible).

I. *Solution by* G. F. WALKER, M.A.; F. BUDD, M.A.; *and others.*

The sides of the parallelogram must touch a confocal whose equation is found to be

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2},$$

and the polar reciprocal of this with respect to the ellipse is $x^2+y^2 = a^2+b^2$ (a result which is evident geometrically); and the tangents to the ellipse at two adjacent points of the parallelogram are therefore at right angles.

Let the points on the ellipse be θ_1, θ_2; then we have

$$\frac{\cos\theta_1\cos\theta_2}{a^2} = -\frac{\sin\theta_1\sin\theta_2}{b^2} = \kappa \text{ suppose.}$$

The equation to any conic through the four points is of the form

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+2\left(\frac{x}{a}\sin\theta_1-\frac{y}{b}\cos\theta_1\right)\left(\frac{x}{a}\sin\theta_2-\frac{y}{b}\cos\theta_2\right)=0.$$

If this cut the ellipse at right angles at the point θ_1, we have

$$\frac{\cos\theta_1}{a}\left[\lambda\frac{\cos\theta_1}{a}+\frac{\sin\theta_1}{a}\sin(\theta_2-\theta_1)\right]+\frac{\sin\theta_1}{b}\left[\lambda\frac{\sin\theta_1}{b}-\frac{\cos\theta_1}{b}\sin(\theta_2-\theta_1)\right]=0$$

$$\therefore\quad \lambda\left(\frac{\sin^2\theta_1}{b^2}+\frac{\cos^2\theta_1}{a^2}\right)=\sin\theta_1\cos\theta_1\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin(\theta_2-\theta_1),$$

$$\lambda\sin\theta_2\cos\theta_2\left(\frac{\sin^2\theta_1}{b^2}+\frac{\cos^2\theta_1}{a^2}\right)$$

$$=\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin(\theta_2-\theta_1),\quad\therefore\ \lambda\kappa\sin(\theta_2-\theta_1),$$

$$=-\kappa^2(a^2-b^2)\sin(\theta_2-\theta_1),\quad\therefore\ \lambda=-\kappa(a^2-b^2).$$

This is symmetrical with respect to θ_1 and θ_2, and therefore the conic cuts the given conic at right angles at θ_2 also.

The equation may be written

$$-\kappa(a^2-b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\left(\frac{a^2}{b^2}y^2-\frac{b^2}{a^2}x^2\right)\kappa-\frac{xy}{ab}\sin(\theta_1+\theta_2)=0,$$

which passes through the foci of the given conic.

II. *Solution by the* Proposer.

The sides of a parallelogram of minimum perimeter inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ must touch a confocal ellipse $\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1$; when λ will be determined by the equation

$$\frac{a^2-\lambda}{a^2}+\frac{b^2-\lambda}{b^2}=1,\quad\text{or}\ \lambda=\frac{a^2b^2}{a^2+b^2};$$

and the sides will touch the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$

The equation of two parallel sides may then be taken to be

$$\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2=\frac{a^2\cos^2\theta+b^2\sin\theta}{a^2+b^2};$$

and the equation of a conic through the corners of the parallelogram is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+k\left\{\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2-\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2+b^2}\right\}=0.$$

One such conic is then $(k=-2)$

$$-\frac{x^2}{a^2}\cos2\theta+\frac{y^2}{b^2}\cos2\theta-\frac{2xy}{ab}\sin2\theta=1-2\,\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2+b^2},$$

or

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\tan2\theta-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2},$$

which runs at right angles the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at all four common points. For at their common points

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2xy}{ab}\tan 2\theta=\frac{a^2-b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right);$$

and this will be found to coincide with the equation

$$\frac{x}{a^2}\left(\frac{x}{a^2}+\frac{y}{ab}\tan 2\theta\right)+\frac{y}{b^2}\left(\frac{x}{ab}\tan 2\theta-\frac{y}{b^2}\right)=0,$$

which is the condition that the tangents at a common point (x, y) may be at right angles.

This conic passes through the four fixed points

$$y=0,\ x^2=\frac{a^2(a^2-b^2)}{a^2+b^2};\quad x=0,\ y^2=\frac{b^2(b^2+a^2)}{b^2-a^2}.$$

6114. (By E. W. Symons, B.A.)—Prove that, in a spheric triangle,

(i.) $\sec^2 A+\sec^2 B+\sec^2 C>12$, (ii.) $\sec A\sec B\sec C>8$;

and in a plane triangle, the triangles not being obtuse,

(i.) $\tan A+\tan B+\tan C=\tan A\tan B\tan C>3\sqrt{3}$,

(ii.) $\cot A+\cot B+\cot C>\sqrt{3}$,

(iii.) $\tan B\tan C+\tan C\tan A+\tan A\tan B>9$.

Solution by the Rev. T. R. Terry, M.A.; *the* Proposer; *and others.*

In a plane acute-angled triangle,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

and $\tan A+\tan B+\tan C>3(\tan A\tan B\tan C)^{\frac{1}{3}}$,

therefore $\tan A+\tan B+\tan C=\tan A\tan B\tan C>3\sqrt{3}$ ..... (1),

Again, since $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$,

therefore $(\cot A+\cot B+\cot C)^2=3+\cot^2 A+\cot^2 B+\cot^2 C$
$-\cot A\cot B-\cot B\cot C-\cot C\cot A$,

therefore $\cot A+\cot B+\cot C>\sqrt{3}$ ..... (2).

From (1) and (2), $\tan A\tan B+\tan B\tan C+\tan C\tan A>9$,

also $\sec^2 A+\sec^2 B+\sec^2 C>3+3(\tan A\tan B\tan C)^{\frac{2}{3}}$, *i.e.* >12,

and $\sec^2 A\sec^2 B\sec^2 C=1+(\tan^2 A+\ldots)+(\tan^2 A\tan^2 B+\ldots)$
$+\tan^2 A\tan^2 B\tan^2 C$, *i.e.* >64,

therefore $\sec A\sec B\sec C>8$.

Now, since these last two expressions increase with A, B, C, and as a plane triangle can always be drawn whose angles have the same ratio to one another as the angles of a spheric triangle, we see that, if the last two inequalities are true for the plane triangle, they are *a fortiori* true for the spherical one.

6475. (See Vol. XXXV., p. 21). II. *Solution by* W. J. C. SHARP, M.A.

If $dy \div \{\phi(y, 1)\}^{\frac{1}{3}}$ be transformed by the proposed substitution, it becomes $(U'V - UV')\,dx \div \{\phi(U, V)\}^{\frac{1}{3}}$,

and $\phi(U, V)$ may be determined so as to have four square factors

$$(x-\alpha)^2,\ (x-\beta)^2,\ (x-\gamma)^2,\ \text{and}\ (x-\delta)^2.$$

The conditions for this will determine the ratios of the coefficients of $\phi(y, 1)$; and the product of $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ will divide $U'V - UV'$, which is itself of the fourth degree in x (see CAYLEY'S *Elliptic Functions*, p. 163); therefore

$$\frac{(U'V - UV')\,dx}{\{\phi(U, V)\}^{\frac{1}{3}}} = \frac{dx}{M\{f(x)\}^{\frac{1}{3}}},$$

where $f(x)$ is the quartic function of x, which is obtained by dividing $\phi(U, V)$ by $(x-\alpha)^2(x-\beta)^2(x-\gamma)^2(x-\delta)^2$.

[This theorem shows that an elliptic integral can always be found which will be transformable by any given transformation of the third order. It appears from the above, however, that it is not possible to find such an integral for a given transformation of a higher order.]

5925. (By E. W. SYMONS, M.A.)—If A', B', C' be any three points on the edges OA, OB, OC of a tetrahedron; prove that the cosine of the angle between the planes ABC, A'B'C' is

$$\{\Delta_1\Delta_1' + \Delta_2\Delta_2' + \Delta_3\Delta_3' - (\Delta_2\Delta_3' + \Delta_2'\Delta_3)\cos A' - \ldots\} \div \Delta\Delta',$$

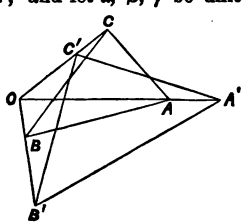
where Δ is the area of the triangle ABC, Δ_1 of OBC, ... &c., and A' the angle between the planes OBC, OCA, &c.

Solution by the PROPOSER; A. L. SELBY, M.A.; *and others.*

Let $OA = a$, $OB = b$, $OC = c$, $OA' = a'$, ...; and let α, β, γ be unit vectors along the axes; then

$$(AB) = b\beta - a\alpha, \quad (AC) = c\gamma - a\alpha,$$

and $\dfrac{V.[(AB).(AC)]}{2\Delta}$ is unit vector perpendicular to plane ABC; so $\dfrac{V[(A'B').(A'C')]}{2\Delta'}$ is unit vector perpendicular to plane A'B'C', and



$$\pm 4\Delta\Delta'.\cos\theta = S.[V.(AB)(AC)][V.(A'B').(A'C')],$$

$\equiv$ [by well known formulæ, see KELLAND's *Quaternions*, Art. 70, (16)]

$S.[(AB).(A'C')].S[(AC).(A'B')]-S[(AB).(A'B')].S[(AC).(A'C')]$

$= S(b\beta-a\alpha)(c'\gamma-a'\alpha).S(c\gamma-a\alpha)(b'\beta-a'\alpha)$

$\quad -S(b\beta-a\alpha)(b'\beta-a'\alpha).S(c\gamma-a\alpha)(c'\gamma-a'\alpha)$

$\equiv (bc'\cos\lambda-c'a\cos\mu-a'b\cos\nu+aa')(b'c\cos\lambda-ca'\cos\mu-ab'\cos\nu+aa')$

$\quad -[aa'+bb'-(ab'+a'b).\cos\nu][aa'+cc'-(ac'+a'c).\cos\mu]$

$\equiv bc.b'c'.\sin^2\lambda+\ldots+\ldots+aa'(bc'+b'c)(\cos\mu\cos\nu-\cos\lambda)+\ldots$

$\equiv 4.\Delta_1.\Delta_1'+\ldots-4(\Delta_2\Delta_3'+\Delta_2'\Delta_3)\cos A$; therefore, &c.

6352. (By W. H. WALENN, Mem. Phys. Soc.)—Check the calculation

$$V' = \frac{1}{3}.\frac{600\times20-384\times16}{20-16}\times36 = 17568$$

by casting out the elevens, the form being preserved, and no quotients of the divisor 11 being known or used in the process.

Solution by the PROPOSER.

The principles that apply to the solution of this question are treated of in the following papers upon unitation, published in the *Philosophical Magazine*:—II. July, 1873; III. May, 1875; VII. Nov. 1877, (par. 27); "Checking calculations," Jan. 1880; and IX. Feb. 1880, (par. 39).

The subtraction of the sum of the digits in odd places, from the sum of the digits in even places, applied to each value or datum, in the same equational form gives $U_{11}V' = U_{11}\left\{\frac{1}{3}.\frac{6\times9-10\times5}{9-5}\times3\right\}$,

writing $U_{11}V'$ for "unitate of V' to the base 11," or "the remainder to the division of V' by 11," regarded as a separate function that expresses the number of units that a given number (V') is in excess of being exactly divisible by 11.

All values of $U_{11}x^{-1}$, excepting that of $U_{11}11^{-1}$ and (in general) $U_{11}(11m)^{-1}$, are identical with the series $U_{11}x^9$, as the series $U_{11}x^n$ repeats after every 10 terms. Therefore

$$U_{11}V' = U_{11}\{4.3(10-6)\times3\} = U_{11}\{3(10-6)\} = U_{11}\{3.4\} = 1,$$

and $U_{11}17568 = 1$.

The correctness of V' = 17568 is thus ascertained within a high degree of probability. The value of V' is taken from J. PRYDE's *Practical Mathematics* (CHAMBERS's *Educational Course*), p. 222.

6424. (By J. W. RUSSELL, M.A.)—In any triangle, prove that

$$\tfrac{27}{4} > (\sin A+\sin B+\sin C)^2 > 6\sqrt{3}\sin A\sin B\sin C.$$

Solution by H. L. ORCHARD, M.A.; D. EDWARDES; *and others.*

Let r_1, r_2, r_3 be the escribed radii; then we have

$$(r_1-r_2)^2+(r_2-r_3)^2+(r_3-r_1)^2 = 2\{(r+4R)^2-3s^2\},$$

therefore $r+4R > s\sqrt{3}$ and $r < \frac{1}{4}R$,

therefore, *à fortiori*, $\frac{9}{2}R > s\sqrt{3}$, or $27R^2 > 4s$;

that is, $\frac{s}{R} > (\sin A+\sin B+\sin C)^2$.

Again, $(a-b)^2+(b-c^2)+(c-a)^2 = 2(s^2-3r^2-12Rr)$,

therefore $s^2 > 3r(r+4R)$ and $(r < \frac{1}{4}R)$,

therefore $s^2 > 27r^2$, or $s^2 > 3\sqrt{3}\Delta$;

that is, $4s^2 > 3\sqrt{3}\dfrac{abc}{R}$, or $(\sin A+\sin B+\sin C)^2 > 6\sqrt{3}\sin A\sin B\sin C$.

[This is an immediate consequence of the well known theorem, that of all triangles that can be inscribed in a given circle, the equilateral has the greatest perimeter; and of all the circumscribed, the least perimeter,—a theorem whereof the above solution furnishes a trigonometrical proof.]

6242 & 6274. (By Prof. ROSANES.)—(6242) Prove that (1) in the plane of three given conics A, B, C there are three sets of points a, b, c, such that A_b, B_c, C_a are respectively identical with B_a, C_b, A_c; where, generally, P_q denotes the polar of a point q with respect to a conic P; (2) of the three triangles formed respectively by the three points a, the three points b, and the three points c, the first is in perspective with the second, the second with the third, and the third with the first; (3) nine points having the last-named property being given, three conics, related to them in the manner above described, are uniquely determinable.

(6274). The coordinates g_1, g_2, g_3, g_4, g_5, g_6 of a line G being given, those of its conjugate G′, relative to a surface F of the second order, are determined by six equations of the form

$$\rho g_i' = \sum_{s=0}^{s=6} C_{i,s}\,.\,g_s \quad (i = 1, 2, \ldots 6),$$

where the $C_{i,s}$ denote minors of the 2nd degree, of the determinant of F, and ρ a constant factor. According to this, the determination of a *self*-conjugate line leads to a sextic equation in ρ. How is this to be reconciled with the fact that every generator of F is self-conjugate?

Solution by W. J. C. SHARP, M.A.

(6242). If the self-conjugate triangle of B and C be taken as triangle of reference, so that $A \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$,

$$B \equiv a'x^2+b'y^2+c'z^2 = 0, \text{ and } C \equiv a''x^2+b''y^2+c''z^2 = 0,$$

and if a, b, c be (α, β, γ), (x, y, z), (ξ, η, ζ) respectively;

$$\frac{ax+hy+gz}{a'\alpha} = \frac{hx+by+fz}{b'\beta} = \frac{gx+fy+cz}{c'\gamma}, \quad \frac{a'\xi}{a''x} = \frac{b'\eta}{b''y} = \frac{c'\zeta}{c''z},$$

$$\frac{a\xi+h\eta+g\zeta}{a''\alpha} = \frac{h\xi+b\eta+f\zeta}{b''\beta} = \frac{g\xi+f\eta+c\zeta}{c''\gamma};$$

therefore $$\frac{a'\left(\frac{aa''}{a'}x+\frac{hb''}{b'}y+\frac{gc''}{c'}z\right)}{a''(ax+hy+gz)}=\frac{b'\left(\frac{ha''}{a'}x+\frac{bb''}{b'}y+\frac{fc''}{c'}z\right)}{b''(hx+by+fz)}$$
$$=\frac{c'\left(\frac{ga''}{a'}x+\frac{fb''}{b'}y+\frac{cc''}{c'}z\right)}{c''(gx+fy+cz)}\quad\ldots\ldots\ldots(1).$$

Equations which determine three points b, similarly three points c, and three points a, are determined. If $\lambda x+\mu y+\nu z=0$ be the equation to the line joining corresponding b and c points (x_1, y_1, z_1) and (ξ_1, η_1, ζ_1),

$$\lambda:\mu:\nu=\frac{1}{x_1}\left(\frac{b''}{b'}-\frac{c''}{c'}\right):\frac{1}{y_1}\left(\frac{c''}{c'}-\frac{a''}{a'}\right):\frac{1}{z_1}\left(\frac{a''}{a'}-\frac{b''}{b'}\right),$$

where (x_1, y_1, z_1) satisfies (1). Hence it appears that $\frac{g\mu-h\nu}{g-h}=\frac{h\nu-f\lambda}{h-f}$.

Consequently the lines joining corresponding b and c points meet in a point, and the triangles formed by the b and c points are in perspective; and, by symmetry, so are those by the c and a points, and by the a and b points. Again, if the full forms be used for $B=0$ and $C=0$, the equations expressing the identities $A_{b_1}=B_{a_1}$, &c., for the nine-points a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3, will determine ratios of the coefficients of each equation uniquely.

(6274). If (x', y', z'), (x'', y'', z'') be two points on the line whose metrical coordinates are $g_1, g_2, g_3, g_4, g_5, g_6$; these are proportional to

$$x'-x'':y'-y'':z'-z'':y'z''-y''z':z'x''-z''x':x'y'-x''y'.$$

And the equations to the conjugate with respect to

$$F\equiv ax^2+by^2+cz^2+2lyz+2mzx+2nxy+2px+2qy+2rz+d=0$$

are $$F'_1x+F'_2y+F'_3z+F'_4=0,\text{ and }F''_1x+F''_2y+F''_3z+F''_4=0,$$

where F_1, F_2, &c. are used to denote $\frac{dF}{dx}$, $\frac{dF}{dy}$, &c.; therefore

$$g'_1:g'_2:g'_3:g'_4:g'_5:g'_6=F'_2F''_3-F''_2F'_3:F'_3F''_1-F''_3F'_1:F'_1F''_2-F''_1F'_2$$
$$:F''_4F'_1-F'_4F''_1:F''_4F'_2-F'_4F''_2:F'_4F''_3-F'_4F''_3,$$

and $$F'_2F''_3-F''_2F'_3=(nr-qm)g_1+(br-ql)g_2+(rl-qc)g_3$$
$$+(bc-l^2)g_4+(lm-nc)g_5+(nl-bm)g_6,\text{ &c. &c.,}$$

which proves the equation given.

The reason why there is no limitation to the number of self-conjugate lines is, that these equations are not independent. In fact, ρ is a constant, as appears from the form taken by these equations if the centre and axes be taken as origin and axes of coordinates; then

$$\rho g_1=bcg_4,\ \rho g_2=acg_5,\ \rho g_3=abg_6,\ \rho g_4=adg_1,\ \rho g_5=bdg_2,\ \rho g_6=cdg_3,$$

and $\rho^2=abcd=\Delta$. Also, from the general forms, it appears that

$$F'_1g'+F'_2g'_2+F'_3g'_3=0\text{ and }F'_1g'_1+F''_2g'_2+F''_3g'_3=0,$$

and for a self-conjugate line $F'_1g_1+F'_2g_2+F'_3g_3=0$; and if one point is on the surface, all points are so, and the line is a generator.

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